

CURVED STRING TOPOLOGY AND TANGENTIAL FUKAYA CATEGORIES

II

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1. INTRODUCTION

In this thesis, we construct new examples of two-dimensional topological quantum field theories (TQFTs) over the prop $C_*(\overline{\mathcal{M}}_{g,n})$. Our primary methods are algebraic: we make use of the well known theorem of Kontsevich and Soibelman [KonSoi] that

Given a compact and smooth $\mathbb{Z}/2\mathbb{Z}$ graded Calabi-Yau A_∞ algebra B for which the Hodge to De-Rham spectral sequence degenerates, a choice of splitting for this spectral sequence gives rise to a TQFT. In section 2 of this thesis we will recall these notions and results in more detail.

The most basic example of such a category is the (dg-enhanced) derived category of quasicoherent sheaves $QCoh(\mathcal{X})$ on a compact and smooth Calabi-Yau variety. This category satisfies all of the above conditons, and the resulting field theory is known as the *B-model* for this Calabi-Yau variety. Homological Mirror Symmetry [Kon] predicts that the associated TQFT is expected to be equivalent to Gromov-Witten TQFT on the mirror CY variety \mathcal{X}^\vee .

Now consider \mathcal{Y} to be a smooth but non-compact Calabi-Yau variety. Then $QCoh(\mathcal{Y})$ is a non-compact Calabi-Yau category, and by a modified version of the theorem of Kontsevich and Soibelman, we can get a so-called *positive-output* TQFT. The Landau-Ginzburg model uses deformation theory to compactify these theories by deforming the above category by a superpotential w , which is an algebraic function with a proper critical set. Recent work [Pre, LinPom] shows that this gives rise to a TQFT. For more on curved algebras, see section 3.

Section 2 of the present paper recalls that a similar situation occurs in topology. Namely, there is a positive output TQFT called *string topology* for a compact oriented manifold \mathcal{Q} associated to the dg-category of dg-modules $mod(C_*(\Omega\mathcal{Q}))$ over the dg algebra $C_*(\Omega\mathcal{Q})$ [Lur], where $\Omega\mathcal{Q}$ denotes the based loop space of \mathcal{Q} at some arbitrary point, pt . Throughout this thesis, all coefficients are taken to be \mathbb{C} , the field of complex numbers. As we explain below, this category is a smooth but not compact category. The relationship with string topology is revealed by the following calculation for the Hochschild homology:

$$\mathbf{HH}_*(C_*(\Omega\mathcal{Q})) \cong C_*(\mathcal{L}\mathcal{Q})$$

There is also a natural compact Calabi-Yau category associated to such a manifold, the category of modules over $C^*(\mathcal{Q})$, which however is not smooth. Such categories give rise to TQFT's with *positive-input*. When \mathcal{Q} is simply connected, these two algebras are related via Koszul duality. Namely, the inclusion $pt \rightarrow \mathcal{Q}$, induces a module structure:

$$C^*(\mathcal{Q}) \rightarrow \mathbb{C}$$

The vector space \mathbb{C} can also be thought of as a module over $C_*(\Omega\mathcal{Q})$ by regarding it as the trivial local system. As discussed in [BluCohTel], the following isomorphisms hold:

$$\mathbb{R}Hom_{C^*(\mathcal{Q})}(\mathbb{C}, \mathbb{C}) \cong C_*(\Omega\mathcal{Q})$$

$$C^*(\mathcal{Q}) \cong \mathbb{R}Hom_{C_*(\Omega\mathcal{Q})}(\mathbb{C}, \mathbb{C})$$

and in fact this gives rise to fully faithful functors:

$$perf(C_*(\Omega\mathcal{Q}) \rightarrow mod(C^*(\mathcal{Q})))$$

and

$$perf(C^*(\mathcal{Q})^{op} \rightarrow mod(C_*(\Omega\mathcal{Q})^{op}))$$

Here $perf(C_*(\Omega\mathcal{Q})$ or $perf(C^*(\mathcal{Q})^{op})$ denotes the subcategory of perfect modules, which is defined for the reader below. Nevertheless, \mathbb{C} is not a compact generator in the category $mod(C_*(\Omega\mathcal{Q}))$ which means that Koszul duality does not give rise to an equivalence of the full derived categories. Following [Abou], Section 1.1 also reviews the relationship between string topology and the Fukaya category of $T^*\mathcal{Q}$, which provides a geometric way of thinking about this Koszul duality.

The case where \mathcal{Q} is $T^n = S^1 \times S^1 \times \dots \times S^1$, served as motivation for the present work. Dyckerhoff [Dyc] proved the following theorem:

Theorem 1.1. *Let w be a function on $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ with isolated singularities. The object \mathbb{C} is a compact generator for $MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)$. Otherwise stated, $Hom_{MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)}(\mathbb{C}, -)$ defines an equivalence of categories:*

$$MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w) \rightarrow mod(Hom_{MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)}(\mathbb{C}, \mathbb{C}))$$

Here $MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)$ denotes the category of matrix factorizations, whose definition occupies much of section 3. The relationship between this theorem and the previous discussion is that $C_*(\Omega T^n)$ is isomorphic to $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_n, z_n^{-1}]$, the Laurent polynomial ring in several variables. As $T^n = S^1 \times S^1 \times \dots \times S^1$ is not simply connected, we complete at the augmentation ideal of this ring to obtain $\mathbb{C}[[x_1, x_2, \dots, x_n]]$. In such cases, $MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)$ defines a quantum field theory. This result can be viewed as a deformed Koszul duality in the sense that $Hom_{MF(\mathbb{C}[[x_1, x_2, \dots, x_n]], w)}(\mathbb{C}, \mathbb{C}) \cong H^*(T^n)$ with a deformed A_∞ structure :

$$m_\ell : H^*(T^n)^{\otimes \ell} \rightarrow H^*(T^n)$$

the coefficients of which can be derived from w in a direct manner [Dyc, Efi].

In this thesis, we will consider simply connected manifolds \mathcal{Q} whose minimal models are pure Sullivan algebras, which are generalizations of complete intersection rings (see section 3 for the precise definition). Section 4 of our paper makes precise and then gives an answer to the following question:

Question 1.2. *If $C^*(\mathcal{Q})$ is a pure Sullivan algebra and given an element $w \in Z(C_*(\Omega\mathcal{Q}))$, when is \mathbb{C} a compact generator of $MF(C_*(\Omega\mathcal{Q}), w)$ defining an equivalence with $\text{mod}(H^*(C^*(\mathcal{Q})), m_\ell)$?*

In section 5, we make the Hochschild cohomology of $MF(C_*(\Omega\mathcal{Q}), w)$ explicit, prove that the deformed category is still Calabi-Yau and deduce the degeneration of the aforementioned Hodge-de Rham spectral sequence. We will examine our condition in the special case that the differential of our pure Sullivan algebra is quadratic. In section 6, we give some comments on the pure Sullivan condition.

As mentioned earlier, morally, one can think of a potential w as “compactifying” the field theory. In section 7 of our paper, inspired by a program of [Sei], we explain how the simplest of our theories, such as when $\mathcal{Q} = \mathbb{C}P^n$ or S^n , arises by geometrically compactifying the cotangent bundles $T^*\mathbb{C}P^n$ and T^*S^n inside of a certain root stack. The definition of the root stack is explained at the beginning of section 7. A slightly more precise statement of our result is that we realize our category $MF(C_*(\Omega\mathcal{Q}), w)$ as being equivalent to the subcategory of the Fukaya category of the root stack generated by the zero section.

Section 8 discusses how the results of section 7 fit into constructions in Symplectic Field Theory. Section 9 discusses the above compactifications from the point of view of Homological Mirror Symmetry and SYZ fibrations, which allows us to refine the equivalence in the previous section for some low dimensional examples.

Section 10 rounds out this thesis by again using SYZ fibrations to generalize the above results to A_n plumbings of cotangent bundles of S^2 . We begin by proving homological mirror symmetry for these plumbings. Then we move on to studying the curved deformations of their wrapped Fukaya categories.

The author would like to thank his adviser Constantin Teleman for sharing his vision of mathematics for many years and for suggesting the general direction of this research. Denis Auroux taught the author about mirror symmetry and symplectic geometry which have played an unexpectedly large role in this work. He also read an early draft of this work and made many suggestions to improve the exposition. The author would also like to thank Mohammed Abouzaid and Yakov Eliashberg for inspiring discussions about symplectic geometry and Anatoly Preygel for teaching him about curved dg-algebras.

2. BACKGROUND AND ALGEBRAIC SETUP

All gradings referred to below will follow homological grading conventions. Given a pre-triangulated dg-category \mathcal{C} , we denote its associated triangulated category by $[\mathcal{C}]$. For invariants derived from these categories, such as $\mathbf{HH}^*(\mathcal{C})$ or $\mathbf{HH}_*(\mathcal{C})$, bold font will be used when it is important that the construction be carried out at the chain level. Any use of functors such as Hom or \otimes is always assumed to be derived.

Recall that a dg-module (or A_∞ -module) N over a dg-algebra (or A_∞ -algebra) \mathcal{A} is *perfect* if it is contained in the smallest idempotent-closed triangulated subcategory of $[mod(\mathcal{A})]$ generated by \mathcal{A} . In general, unless explicitly stated otherwise, we will use the term “generation” to mean what symplectic geometers usually call “split-generation”. That is to say, given a collection of objects \mathbf{O}_i in a triangulated category \mathcal{C} , the subcategory generated by \mathbf{O}_i will be the smallest idempotent-closed triangulated subcategory containing \mathbf{O}_i .

Definition 2.1. A dg-algebra \mathcal{A} over \mathbb{C} is *compact* if \mathcal{A} is perfect as a \mathbb{C} module (in this special case this simply says that \mathcal{A} is equivalent to a finite dimensional vector space). A dg-algebra \mathcal{A} is *smooth* if \mathcal{A} is perfect as an $\mathcal{A} - \mathcal{A}$ bimodule.

A very useful criterion for smoothness is given by the notion of finite-type of Toën and Vaquie [ToëVaq].

Definition 2.2. A dg-algebra \mathcal{A} is of finite type if it is a homotopy retract in the homotopy category of dg-algebras of a free algebra $(\mathbb{C}\langle v_1, v_2, \dots, v_n \rangle, d)$ with $dv_j \in \mathbb{C}\langle v_1, v_2, \dots, v_{j-1} \rangle$

Lemma 2.3. *If \mathcal{A} is of finite type then \mathcal{A} is smooth. The converse is also true if \mathcal{A} is assumed to be compact.*

Lemma 2.4. *With the notation of the previous section, the dg-algebra $C_*(\Omega\mathcal{Q})$ is smooth.*

In the simply connected case, this follows from the classical Adams-Hilton construction [AdaHil] and the above theorem of Toën-Vaquie. Consider a cellular model for \mathcal{Q} with cells in dimension $\leq \dim(\mathcal{Q})$ and no 1-cells. Let \mathcal{A} denote the tensor algebra generated by variables e_i^b , $\deg(e_i^b) = b-1$, for all n and i , where e_i^b are in bijection with cells of dimension b in the cell decomposition.

Let A_b denote the algebra generated by cells of dimension $\leq b$. For a given cell, $d(e_i^b) = z$, where z is defined as the pushforward the canonical class in $H_{b-2}(\Omega S^{b-1})$ under the attaching map $f : S^{b-1} \rightarrow \mathcal{Q}$. Thus, we can see that the differential d maps $d : A_b \rightarrow A_{b-1}$ and that the algebra is of finite type. The theorem remains true in the non-simply connected case, but the proof is more complicated [Kon2].

Definition 2.5. A dg-algebra \mathcal{A} is Calabi-Yau of dimension n if $Hom_{\mathbb{C}}(\mathcal{A}, \mathbb{C}) \cong \mathcal{A}[-n]$ as \mathcal{A} - \mathcal{A} bimodules.

Example 2.6. Given a manifold \mathcal{Q} , the algebra $C^*(\mathcal{Q})$ is Calabi-Yau by Poincare duality.

Definition 2.7. A smooth dg-algebra \mathcal{A} is non-compact Calabi-Yau if

$$\mathrm{Hom}_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e) \cong \mathcal{A}[n]$$

as \mathcal{A} - \mathcal{A} bimodules.

Example 2.8. Given a manifold \mathcal{Q} , we can again let $\mathcal{A} = C_*(\Omega\mathcal{Q})$. This is a non-compact Calabi-Yau. To see this, note that by smoothness we have that:

$$\mathrm{Hom}_{\mathcal{A}^e}(\mathrm{Hom}_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e), \mathcal{A}[n]) \cong \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}[n]$$

As noted above we have that:

$$\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}[n] \cong C_*(\mathcal{L}\mathcal{Q})[n]$$

The fundamental class of \mathcal{Q} provides the desired isomorphism

$$\mathrm{Hom}_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e) \cong \mathcal{A}[n]$$

Next we recall a tiny bit about how the duality between $C^*(\mathcal{Q})$ and $C_*(\Omega\mathcal{Q})$ for compact simply connected manifolds is reflected in the symplectic geometry of their cotangent bundles. Consider $T^*\mathcal{Q}$ with its standard symplectic form $d\theta$. Let b , the background class, be the class in $H^*(T^*\mathcal{Q}, \mathbb{Z}_2)$ given by the pullback of the second Stiefel-Whitney class of \mathcal{Q} . The classical Fukaya category, $\mathrm{Fuk}(T^*\mathcal{Q}, b)$, consists of (twisted complexes of) compact exact Lagrangian submanifolds \mathcal{L} such that the restriction of b to \mathcal{L} is $w_2(\mathcal{L})$. For any two Lagrangian submanifolds \mathcal{L}_1 and \mathcal{L}_2 , their morphisms are defined by the Floer homology groups with coefficients in \mathbb{Q} , $HF^*(\mathcal{L}_1, \mathcal{L}_2)$ [Sei2]. The zero section \mathcal{Q} defines such an object. We have the following description of its endomorphisms:

$$HF^*(\mathcal{Q}, \mathcal{Q}) \cong C^*(\mathcal{Q})$$

For Liouville symplectic manifolds such as $T^*\mathcal{Q}$, it is convenient to consider a version of the Fukaya category, known as the *wrapped Fukaya category*, $WFuk(T^*\mathcal{Q})$, which allows us to incorporate non-compact Lagrangians into the Fukaya category. An important example of such an object is the cotangent fibre to a point q , denoted as T_q . For its definition see [Abou]. One very important property of the wrapped Fukaya category is that we have a natural fully faithful functor:

$$i : \mathrm{Fuk}(T^*\mathcal{Q}) \rightarrow WFuk(T^*\mathcal{Q})$$

Abouzaid proves the following theorem:

Theorem 2.9. *The cotangent fibre strongly generates the wrapped Fukaya category of $T^*\mathcal{Q}$ with background class $b \in H^*(T^*\mathcal{Q}, \mathbb{Z}_2)$ given by the pullback of the second Stiefel-Whitney class of \mathcal{Q} . The triangulated closure of the wrapped Fukaya category is equivalent to the category $\mathrm{perf}(C_*(\Omega\mathcal{Q}))$.*

The second sentence follows from the first because of the following description of the endomorphisms of the cotangent fibre:

$$WHF^*(T_q, T_q) \cong C_*(\Omega\mathcal{Q})$$

3. PURE SULLIVAN ALGEBRAS AND CURVED ALGEBRAS

We consider Pure Sullivan dg-algebras \mathcal{B} of the form:

$$(\wedge V, d) = (\mathbb{C}[x_1, \dots, x_n] \otimes \bigwedge(\beta_1, \dots, \beta_m), d(\beta_i) = f_i(x_1, \dots, x_n), d(x_j) = 0)$$

where the $\deg(x_i)$ are even and negative, the functions f_i have no linear term, and the $\deg(\beta_i)$ are odd > 1 . We further assume that $\dim(H^*(\mathcal{B})) < \infty$.

One of the underlying ideas of Chevalley-Eilenberg theory is that such algebras determine an L_∞ model \mathfrak{g} for \mathcal{B} . We can define an algebra \mathcal{A} which is the universal enveloping algebra $U\mathfrak{g}$ of these Lie algebras. We briefly explain certain ideas from rational homotopy theory which will be used extensively below. For more details, the reader is encouraged to consult [FelHalTho]. To a simply connected space of finite type, \mathcal{M} , one can assign an L_∞ algebra $\mathfrak{g} = \pi_*(\Omega(\mathcal{M})) \otimes \mathbb{Q}$, with Whitehead-Samelson bracket. To recover $C^*(\mathcal{M})$, one considers the Chevalley-Eilenberg complex $C^*(\mathfrak{g})$, a canonical complex that computes Lie-algebra cohomology with coefficients in the trivial module. We have a quasi-isomorphism,

$$C^*(\mathfrak{g}) \rightarrow C^*(\mathcal{M})$$

Furthermore, one can show that $U\mathfrak{g}$ is then quasi-isomorphic to $C_*(\Omega(\mathcal{M}))$. To go in the other direction, the theory of rational homotopy allows us to assign a space, \mathcal{M} , well defined up to rational homotopy equivalence, to a 1-connected dga or equivalently a connected dg-Lie algebras of finite type.

In the case of pure Sullivan algebras \mathcal{B} , there is a concrete description of the universal enveloping A_∞ algebra \mathcal{A} . Using the homological perturbation lemma, we have an explicit A_∞ model for \mathcal{A} of the form

$$(Sym(\mathfrak{g}_{even}) \otimes \Lambda(\mathfrak{g}_{odd}), m_n)$$

A formula for the higher multiplications appears in section 3 of [Bar]. For our purposes, we note the following facts. First, the strict morphism of the abelian Lie algebra $\pi_{even}(\Omega(Q)) \rightarrow \mathfrak{g}$ corresponds to the inclusion of $Sym(\mathfrak{g}_{even}) \cong \mathbb{C}[u_1, \dots, u_m] \rightarrow \mathcal{A}$. The higher multiplications m_n are multi-linear in these variables for $n \geq 3$. Finally, we have that the A_∞ algebra is strictly unital and the augmentation $U\mathfrak{g} \rightarrow \mathbb{C}$ defined by killing $\mathfrak{g}U\mathfrak{g}$ is also a strict morphism.

The reader should be warned that in the presence of quadratic terms in the f_i , the above identification with $Sym(\mathfrak{g}_{even}) \otimes \Lambda(\mathfrak{g}_{odd})$ is only an identification of vector spaces. In other words, there can

be a non-trivial Lie bracket $B : \mathfrak{g}_{odd} \otimes \mathfrak{g}_{odd} \rightarrow \mathfrak{g}_{even}$, which means that forgetting higher products, $U\mathfrak{g}$ is a Clifford algebra over $Sym(\mathfrak{g}_{even})$. It also seems worth pointing out that the even variables u_i can be thought of as being Koszul dual to the odd variables β_i . Meanwhile the variables in \mathfrak{g}_{odd} , from here on denoted as e_j , are dual to the even variables x_j above.

Next, we discuss how to define an appropriate category of matrix factorizations. This section adopts the ideas of the foundational work [Pre] to our non-commutative context. For concreteness, let us consider as before the above A_∞ -algebra \mathcal{A} , and an element $w \in \mathbb{C}[u_1, \dots, u_m]$ of degree $2j - 2$. For example, if $M = \mathbb{C}P^n$, we have the following specific model:

$$U\mathfrak{g} = \mathbb{C}[u] \otimes \Lambda(e), m_{n+1}(e, e, e, \dots, e) = u$$

We can then consider potentials of the form $w = u^d$.

We define a variable x of degree $2j - 2$. The element w defines a mapping from

$$w : \mathbb{C}[x] \rightarrow \mathcal{A}$$

and we can consider the A_∞ algebra $\mathcal{A}_0 = (\mathcal{A}[e], de = w)$, where e now has degree $2j - 1$.

Definition 3.1. We define $Pre(MF(\mathcal{A}, w))$, to be the full subcategory of $mod(\mathcal{A}_0)$ consisting of modules which are perfect over \mathcal{A} .

This category is equipped with a natural $\mathbb{C}[[t]]$ (degree $t = -2j$) linear structure which we will now describe.

Remark 3.2. Of course, $\mathbb{C}[[t]]$ as a graded ring is usually denoted $\mathbb{C}[t]$. The notation $\mathbb{C}[[t]]$ is simply to note that it should be treated as a topological ring. For example, given a graded vector space V ,

$$V[[t]]_n = \prod_{k \geq 0} V_{n+2jk}$$

This will be distinct from $V[t]$ if V is not homologically bounded from above.

We begin with the description of the $\mathbb{C}[[t]]$ -linear structure from an abstract point of view and then give more concrete descriptions. We observe that:

$$Pre(MF(\mathcal{A}, w)) \cong RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A}))$$

the category of colimit preserving functors[Toë]. We describe this construction a bit more below. For the reader who is become disoriented with the notation, notice that the $\mathbb{C}[x]$ structure on the right-hand side comes from the above algebra map w .

This category of functors is acted upon the category $RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathbb{C}))$ by convolution. Let α denote a variable of degree $2j - 1$. Then there is an isomorphism:

$$RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathbb{C})) \cong D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$$

Here $D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$ denotes the subcategory category of modules over $\mathbb{C}[\alpha]/\alpha^2$ which are homologically finite over \mathbb{C} . Next we notice that Koszul duality provides an equivalence :

$$D_{fin}(\mathbb{C}[\alpha]/\alpha^2) \cong Perf(\mathbb{C}[[t]])$$

The aforementioned $\mathbb{C}[[t]]$ (degree $t = -2n$) linear structure now arises in view of the natural equivalence between (idempotent complete, pre-triangulated) module categories over $Perf(\mathbb{C}[[t]])$ and ordinary $\mathbb{C}[[t]]$ -linear, (idempotent complete, pre-triangulated) dg categories.

This description of the action of t is relatively obscure and so we now aim to unravel it and make it more concrete. The object \mathbb{C} in $D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$ acts via the identity. The action for the module $\mathbb{C}[\alpha]/\alpha^2$ can be described by considering the composition of the two adjoint-functors:

$$i_* : RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A})) \rightarrow RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}[x]), Perf(\mathcal{A}))$$

$$i^* : RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}[x]), Perf(\mathcal{A})) \rightarrow RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A}))$$

In view of the fact that $i^* \circ i_*(N) \cong \mathbb{C} \otimes_{\mathbb{C}[x]} N$, and the fact that N and \mathbb{C} are perfect over \mathcal{A} and $\mathbb{C}[x]$ respectively, it follows that $i^* \circ i_*(N)$ is perfect over \mathcal{A}_0 .

One can resolve the module \mathbb{C} over $\mathbb{C}[\alpha]/\alpha^2$ by the standard Koszul resolution:

$$\mathbb{C} \cong \bigoplus_k \frac{\mathbb{C}[\alpha]}{\alpha^2} [u^k/k!], du = \alpha$$

Applying this resolution to an object in M in $Pre(MF(\mathcal{A}, w))$, we conclude that:

$$Hom_{Pre(MF(\mathcal{A}, w))}(M, N) = (Hom_{Perf(\mathcal{A})}(M, N)[[t]], d)$$

where

$$d : \phi \rightarrow d_{\mathcal{A}}(\phi) + t(\phi \circ e \wedge + e \wedge \circ \phi)$$

The differential $d_{\mathcal{A}}$ denotes the differential on $Hom_{Perf(\mathcal{A})}(M, N)$. In this equation t acts in the natural way.

The above construction generalizes the construction of the category of singularities for ordinary commutative rings [Orl]. It is natural to ask how this $\mathbb{C}[[t]]$ linear structure arises from deformation theory or how it can be expressed in a way that resembles the usual category of matrix factorizations. The first step is to define a reasonable category of modules for the topologically complete unital, augmented, curved A_{∞} algebra

$$(\mathcal{A}[[t]], tw)$$

The following construction was outlined in [Pom]. We denote by \mathcal{A}_+ the quotient \mathcal{A}/\mathbb{C} . We note that the element tw also defines a Maurer-Cartan element in $\mathbf{HH}^*(\mathcal{A}, \mathcal{A})[[t]]$. Such a Maurer-Cartan solution allows us to twist the differential on

$$(\bigoplus_n \mathcal{A}_+^{\otimes n}[[t]], d_A)$$

by the differential determined by the formula:

$$td_w : a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^{i+1} ta_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes W \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

giving rise to a topologically complete coalgebra:

$$\mathbf{C} = (\bigoplus_n \mathcal{A}_+^{\otimes n}[[t]], d_A + td_w)$$

We can look now at modules over this coalgebra which are topologically free over $\mathbb{C}[[t]]$, topologically cofree as modules over the underlying coalgebra, and are perfect over A when $t = 0$. We denote this category by $comod(\mathbf{C})$.

Lemma 3.3. *The functor $\mathcal{F} : M \rightarrow ((\bigoplus_n \mathcal{A}_+^{\otimes n} \otimes M)[[t]], d_{M/A} + te \wedge)$ defines a fully faithful functor:*

$$Pre(MF(\mathcal{A}, w)) \rightarrow comod(\mathbf{C})$$

Ignoring differentials for ease of notation we have that:

$$Hom(\mathcal{F}(M), \mathcal{F}(N)) = Hom_{\mathbb{C}[[t]]}^{top}((\bigoplus_n \mathcal{A}_+^{\otimes n} \otimes M)[[t]], N[[t]])$$

We can identify this with:

$$Hom_{\mathbb{C}}(\bigoplus_n \mathcal{A}_+^{\otimes n} \otimes M, N)[[t]]$$

The differential on this complex is again the differential $d_A + t(\phi \circ e \wedge + e \wedge \circ \phi)$.

Finally, we define

$$MF(\mathcal{A}, w) = Pre(MF(\mathcal{A}, w)) \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)) \cong comod(\mathbf{C}) \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$$

The fact that $i^* \circ i_*(N)$ is perfect for any object in $Pre(MF(\mathcal{A}, w))$ implies just as in the usual case that

$$[MF(\mathcal{A}, w)] \cong [Pre(MF(\mathcal{A}, w))]/[Perf(\mathcal{A}_0)]$$

It is often convenient to work with the formal Ind-completion $Ind(MF(\mathcal{A}, w))$ which we shall denote by $MF^\infty(\mathcal{A}, w)$.

We have constructed a category of curved modules for a curved A_∞ algebra which arises as a deformation of an uncurved A_∞ algebra. A recent article by Positselski [Pos, Pos2] explains a similar construction of a module category for curved A_∞ algebras \mathcal{A} over complete local rings k where the potential is contained inside $m\mathcal{A}$. This is consistent with the above philosophy that it is necessary to treat $\mathbb{C}[[t]]$ as a topological ring to construct a good category of modules. There is no good general construction of a module category over a curved A_∞ algebra algebra.

To illustrate this point, it might be useful to consider the case of ordinary matrix factorizations, namely pairs (R, w) where R is a commutative ring as above. If one considers it as a two-periodic curved A_∞ -algebra, various authors [CalTu] have noted that the category of comodules over the two-periodic bar algebra:

$$\bigoplus_n R^{\otimes n}, d_R + d_w$$

is always zero. Thus it is important to consider the bar complex over $\mathbb{C}[[t]]$, calculate the corresponding Hom-sets and then invert t . As an example, consider the case when the function w has isolated singularities. The naive Hochschild complex

$$(\prod Hom_{\mathbb{C}}(R^{n+1}, R), d_{Hoch} + [w, \cdot])$$

is always zero. However we have a quasi-isomorphism as two periodic complexes:

$$(\prod Hom_{\mathbb{C}}(R^{n+1}, R)((t)), d_{Hoch} + t[w, \cdot]) \cong (\bigoplus Hom_{\mathbb{C}}(R^{n+1}, R), d_{Hoch} + [w, \cdot])$$

This latter complex computes the Jacobian ring as one would expect.

4. THE CRITERION FOR GENERATION

In this section we discuss a criterion for smoothness and properness of the category $MF(\mathcal{A}, w)$. To state the criterion, we must consider the category of curved bimodules

$$MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$$

and we define $\mathbf{HH}^*(MF(\mathcal{A}, w))$ to be $Hom_{MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)}(\mathcal{A}, \mathcal{A})$. Using either description of our category, this can be computed explicitly as:

$$\mathbf{HH}^*(MF(\mathcal{A}, w)) \cong (\mathbf{HH}^*(\mathcal{A}, \mathcal{A})((t)), d_{\mathcal{A}} + [tw, \cdot])$$

The following is the analogue of Dyckerhoff's theorem for our situation:

Theorem 4.1. *If $\mathbf{HH}^*(MF(\mathcal{A}, w))$ is finite over $\mathbb{C}((t))$, then $\mathbb{C}((t))$ generates the category $MF(\mathcal{A}, w)$.*

We have an action of $\mathbb{C}[u_1, \dots, u_m]$ on $DSing(\mathcal{A}_0)$ which factors through the complex $\mathbf{HH}^*(MF(\mathcal{A}, w))$. For any u in $\mathbb{C}[u_1, \dots, u_m]$, we let K_u be the diagram

$$\mathbb{C}[u_1, \dots, u_m] \xrightarrow{u} \mathbb{C}[u_1, \dots, u_m]$$

Finally, for the sequence $\bar{u} = (u_1, \dots, u_m)$ we define

$$K_{\bar{u}^p} = \otimes K_{u_i^p}$$

With this notation in hand, we consider the colimit of the diagram:

$$K_{\bar{u}} \rightarrow K_{\bar{u}^2} \rightarrow K_{\bar{u}^3} \dots$$

which we denote by $R\Gamma_m$. For any object \mathbf{O} in $MF(\mathcal{A}, w)$, we have an augmentation

$$R\Gamma_m \otimes_{\mathbb{C}[u_1, \dots, u_m]} \mathbf{O} \rightarrow \mathbf{O} \rightarrow \text{cone}(e)$$

Because the action of $\mathbb{C}[u_1, \dots, u_m]$ factors as above, we see that such an m-equivalence is in fact an equivalence and can conclude that $\text{cone}(e)$ is zero.

Now the objects $K_{\bar{u}^i} \otimes \mathbf{O}$ are in the triangulated subcategory generated by \mathbb{C} because \mathcal{A} is finitely generated as a module over $\mathbb{C}[u_1, \dots, u_m]$. The objects \mathbf{O} are compact in $MF^\infty(\mathcal{A}, w)$ and can be expressed as a colimit of $K_{\bar{u}^i} \otimes \mathbf{O}$. Therefore we can conclude that \mathbf{O} is a direct summand of one of the $K_{\bar{u}^i}(\mathbf{D}) \otimes \mathbf{O}$ generated by \mathbb{C} as well.

Remark 4.2. The same argument goes through if the ideal of the kernel of the above ring homomorphism is I . Namely, in this situation, the category is generated by $\mathbb{C}[u_1, \dots, u_m]/I \otimes_{\mathbb{C}[u_1, \dots, u_m]} \mathcal{A}$. We isolate the case where $\mathbb{C}[u_1, \dots, u_m]/I$ is finite dimensional because it has the most relevance to topological field theories.

To discuss homological smoothness, we must consider the category:

$$RHom_{\mathbb{C}(t)}^c(MF^\infty(\mathcal{A}, w), MF^\infty(\mathcal{A}, w))$$

the category of continuous endofunctors in the sense of [Toë], which we now describe. Given two dg-categories \mathcal{C}_1 and \mathcal{C}_2 , the naive category of dg-functors $\text{Hom}(\mathcal{C}_1, \mathcal{C}_2)$ is not well behaved with respect to quasi-equivalence of dg-categories.

Toën [Toë] proved that there is a model structure on the category of dg-categories where weak equivalences are given by quasi-equivalences and the category $RHom(\mathcal{C}_1, \mathcal{C}_2)$ is a derived functor with respect to this model structure. We have a natural inclusion $RHom^c(\mathcal{C}_1, \mathcal{C}_2) \subset RHom(\mathcal{C}_1, \mathcal{C}_2)$ of all functors which commute with arbitrary colimits.

Toën proves that for if a co-complete dg-category \mathcal{C} has a compact generator \mathbf{O} , and is thus equivalent to the category of modules $\text{mod}(Hom(\mathbf{O}, \mathbf{O})^{op})$, then we have that [Toë]:

$$RHom^c(\mathcal{C}, \mathcal{C}) \cong \text{mod}(Hom(\mathbf{O}, \mathbf{O}) \otimes Hom(\mathbf{O}, \mathbf{O})^{op})$$

Thus we have the following theorem:

Theorem 4.3. $RHom_{\mathbb{C}((t))}^c(MF^\infty(\mathcal{A}, w), MF^\infty(\mathcal{A}, w)) \cong MF^\infty(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$

This follows because \mathbb{C} also generates the category $MF^\infty(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$. Let t_1 be the deformation parameter corresponding to $Pre(MF(\mathcal{A}, w))$ and t_2 be the deformation parameter corresponding to $Pre(MF(\mathcal{A}^{op}, -w))$. We know that $Hom_{\mathcal{A}}(\mathbb{C}, \mathbb{C})$ is homologically bounded above. This implies that the mapping:

$$Hom_{\mathcal{A}}(\mathbb{C}, \mathbb{C})[[t_1]] \otimes_{\mathbb{C}[[t]]} Hom_{\mathcal{A}}(\mathbb{C}, \mathbb{C})[[t_2]] \rightarrow Hom_{\mathcal{A} \otimes \mathcal{A}^{op}}(\mathbb{C}, \mathbb{C})[[t_1, t_2]] \otimes \mathbb{C}[[t]]$$

is an equivalence. We achieve the result by inverting t .

Remark 4.4. One of the downsides of the formalism that we have chosen is that the above theorem will fail almost uniformly if the generating object \mathbf{O} in $MF^\infty(\mathcal{A}, w)$ does not have the property that $Hom(\mathbf{O}, \mathbf{O})$ is homologically bounded from above, essentially because the above mapping will almost never be an equivalence. This situation might be improved by giving a definition of the monoidal category of dg-categories where one makes use of completed tensor products.

5. THE CALABI-YAU PROPERTY, HOCHSCHILD COHOMOLOGY AND THE DEGENERATION CONJECTURE

Next we will show that the Calabi-Yau condition for our category $MF(\mathcal{A}, w)$ follows from \mathcal{A} being non-compact Calabi-Yau. We first briefly recall why \mathcal{A} is non-compact Calabi-Yau. Recall that this means that there is an isomorphism of $\mathcal{A} - \mathcal{A}$ -bimodules

$$RHom_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) \cong \mathcal{A}[n]$$

The dg-algebra \mathcal{B} is rationally elliptic. By results in Chapter [35] of the book [FelHalTho] the algebra $H^*(\mathcal{B})$ is a Poincare duality algebra. Now results in [Cos] prove that the deformation theory of C_∞ algebras and Frobenius C_∞ algebras with a fixed trace coincide. By applying the perturbation lemma and viewing the dg-algebra \mathcal{B} as a deformation of $H^*(\mathcal{B})$, this implies that the Frobenius structure on $H^*(\mathcal{B})$ enhances naturally to a Calabi-Yau structure on \mathcal{B} .

Next we have the following theorem proved in [VDBergh]:

Theorem 5.1. *Let \mathcal{A} be a homologically smooth algebra concentrated in degree ≥ 0 . Then a cyclic A_∞ structure on its Koszul dual algebra gives rise to a non-compact Calabi-Yau structure on \mathcal{A} .*

Now that \mathcal{A} is seen to be Calabi-Yau, we show that this implies the property for $MF(\mathcal{A}, w)$. To prove the Calabi-Yau property for $MF(\mathcal{A}, w)$, we note that we have a relative dualizing functor:

$$\begin{aligned} D : MF(\mathcal{A}, w) &\rightarrow MF(\mathcal{A}^{op}, -w)^{op} \\ M &\rightarrow RHom_{\mathcal{A}}(M, \mathcal{A}) \end{aligned}$$

D is manifestly an equivalence. Note that for any object O in $MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$ we have that

$$Hom_{MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)}(\mathbb{C}, O) \cong Hom_{MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)}(D(O), D(\mathbb{C}))$$

We know that

$$D(\mathbb{C}) \cong \mathbb{C}[n]$$

and because \mathcal{A} is non-compact Calabi-Yau, we have that

$$D(\Delta) \cong \Delta[n]$$

From here we learn that for the diagonal

$$Hom_{MF}(\Delta, \mathbb{C}) \cong Hom_{MF}(\mathbb{C}, \Delta)$$

Next we denote the complex $Hom_{MF(\mathcal{A}, w)}(\mathbb{C}, \mathbb{C})$ by \mathcal{D} . This means that there is an isomorphism of $\mathcal{D} - \mathcal{D}$ -bimodules

$$RHom_{\mathcal{D}^e}(\mathcal{D}, \mathcal{D}^e) \cong \mathcal{D}[n]$$

which implies the Calabi-Yau condition for $MF(\mathcal{A}, w)$.

We can make our condition on finiteness of $\mathbf{HH}^*(MF(\mathcal{A}, w))$ more tractable by considering the deformation theory of the pure Sullivan algebra \mathcal{B} itself. As noted in the introduction, for any simply connected space of finite type, we have fully faithful functors induced by the $C^*(\mathcal{Q}) - C_*(\Omega\mathcal{Q})$ bimodule \mathbb{C} . It then follows from a result of Keller [Kel] that for such a fully faithful functor there is a canonical equivalence in the homotopy category of $B(\infty)$ algebras:

$$\mathbf{HH}^*(C^*(\mathcal{Q}), C^*(\mathcal{Q})) \cong \mathbf{HH}^*(C_*(\Omega\mathcal{Q}), C_*(\Omega\mathcal{Q}))$$

In particular these two Koszul dual algebras have equivalent formal deformation theories. Suppose that, more generally, we consider a commutative algebra free-graded commutative model $(\bigwedge V, d)$ where V is a finite dimensional graded vector space. There is a very explicit complex quasi-isomorphic as a dg-Lie algebra to $\mathbf{HH}^*((\bigwedge V, d), (\bigwedge V, d))$. Recall that $T^{poly}(V)$ is the Lie-algebra of polyvector fields on $\bigwedge V$ with Schouten bracket. Part of Kontsevich's formality theorem says that the HKR map:

$$T^{poly}(V) \rightarrow \mathbf{HH}^*(\bigwedge V)$$

is the first Taylor coefficient in an L_∞ quasi-isomorphism between the two.

We can think of the derivation d as corresponding to a vector-field v . It follows from a spectral sequence argument that the HKR map gives a quasi-isomorphism:

$$(T^{poly}(V), [v, -]) \rightarrow \mathbf{HH}^*((\bigwedge V, d), (\bigwedge V, d))$$

Lemma 5.2. *This map can be corrected to an L_∞ quasi-isomorphism. In the case of a pure Sullivan algebra, the first Taylor coefficient agrees with the HKR map.*

Denote by f_n the Taylor coefficients of the Kontsevich formality morphism. We consider a modified L_∞ map

$$\widetilde{f}_n : (T^{poly}(V), [v, -]) \rightarrow \mathbf{HH}^*(\bigwedge(V, d))$$

given by the formula

$$\widetilde{f}_n(x_1, \dots, x_n) = \sum 1/k! f_{k+n}(v, v, \dots, v, x_1, x_2, \dots, x_n)$$

This sum, seemingly a sum consisting of infinitely many terms, makes sense in this case for the following reason: for all $\gamma_1, \dots, \gamma_m$ in $T^{poly}(V)$,

$$f_{n+k}(v, v, \dots, v, x_1, x_2, \dots, x_n)(\gamma_1, \dots, \gamma_m) \text{ vanishes unless } 2(n+k) + m - 2 = k + \sum (|x_i|)$$

It is easy to check that this defines an L_∞ map between the two complexes in question. We must further convince ourselves that the map \widetilde{f}_1 remains a quasi-isomorphism. By the above equation, we know that for any $x \in T^{poly}(V)$ such that $f_1(x) \in HH^m$, we have that

$$\widetilde{f}_1(x) = f_1(x) + \alpha$$

where $\alpha \in HH^{<m}$. Assuming the map is surjective onto $HH^{<m}$, we learn by induction that the map is surjective onto $HH^{\leq m}$ as well. Injectivity of the map on homology is also clear.

In general the formula for our map \widetilde{f}_1 can be computed explicitly from work of [Cal] but is very complicated, the coefficients of the map being given in terms of Bernoulli numbers. In the case of interest, we actually know more, namely, we have the following:

Lemma 5.3. *If the dg algebra is pure Sullivan the map \widetilde{f}_1 agrees with the map f_1 .*

Following [Ca], we denote our coordinates by u_k , and write $v = \sum v_i \partial_i$. There is a matrix valued one form given by

$$\Gamma_{ji} = \sum_k \partial_i \partial_k v_j du_k$$

and define

$$\theta = \sum_{n>0} c_n i_{tr}(\Gamma^n)$$

where the c_n are certain rational coefficients.

Calaque proves that

$$\widetilde{f}_1 = f_1(e^\theta)$$

Now if $v = \sum f(x_1, \dots, x_n) d/de_i$ as above, then the matrix is strictly upper triangular, the trace of any power of it is therefore zero, and thus $\widetilde{f}_1 = f_1$.

In the pure Sullivan case, potentials tw in $\mathbf{HH}^*(\mathcal{A}, \mathcal{A})[[t]]$ correspond to odd-polyvector fields

$$tw(d/de_1, d/de_2, \dots, d/de_m) \in T^{poly}(B)[[t]]$$

The polyvector-field $v = \sum f(x_1, \dots, x_n) d/de_i$ in $T^{poly}(\wedge(V^*[1]))$ gives rise to a deformation of $(\wedge V^*[1])$. The general theory of deformations of Koszul dual algebras [CalFelFerRos] shows that this deformation is isomorphic to \mathcal{A} . Since the Kontsevich formality map has the property that

$$f_{n+1}(w, x_1, x_2, \dots, x_n) = 0, n > 1$$

this proves that tw corresponds to the polyvector field we have claimed.

After passing to the generic fiber, the Hochschild cohomology is given by :

$$(T^{poly}(V)((t)), [v + tw(d/de_1, \dots, d/de_m),])$$

Definition 5.4. By analogy with the case of ordinary matrix factorizations, we will say that w has an *isolated singularity* if the homology of this complex is finite dimensional.

Next we discuss the degeneration conjecture. For (\mathcal{A}, w) as above with isolated singularities, we can see from the formula for Hochschild cohomology that $HH^*(\mathcal{A}, w)$ will always be concentrated in even degree. To see this, note that if we replace the variables d/de_i by u_i then we have a sequence of $\mathbb{Z}/2j\mathbb{Z}$ graded complexes

$$(T^{poly}(C[[u_1, u_2, \dots]][x_1, \dots, x_n]), [w + \sum f_i u_i,]) \subset HH^*(\mathcal{A}, w)$$

and

$$HH^*(\mathcal{A}, w) \subset (T^{poly}(C[x_1, \dots, x_n][[u_1, u_2, \dots, u_m]]), [w + \sum f_i u_i,])$$

In the special case when the cohomology of the complex is finite dimensional this implies that $w + \sum f_i u_i$ has isolated singularities. The degeneration conjecture is then automatic because the Hochschild cohomology is automatically concentrated in even degrees. As the category $MF(\mathcal{A}, w)$ is Calabi-Yau, the Hochschild homology will all be concentrated in the either even or odd degree (depending upon the parity of the Calabi-Yau structure) and the degeneration conjecture thus follows for these algebras without any additional work.

Example 5.5. For $\prod S^{2n_j}$ the condition that w has an isolated singularity is similar to the usual Jacobian condition and states that $\mathbb{C}[u_1, \dots, u_m]/(u_i dw/du_i)$ be finite dimensional. The proof follows from the more general statement below.

More generally, we again discuss the case when \mathfrak{g} is formal. In this case, recall that \mathfrak{g} is determined by a bilinear form:

$$B : \mathfrak{g}_{odd} \otimes \mathfrak{g}_{odd} \rightarrow \mathfrak{g}_{even}$$

and $U\mathfrak{g}$ is a graded Clifford algebra over $\mathbb{C}[u_1, \dots, u_m]$. We let D_k be the closed subvariety of $\mathbb{C}[u_1, \dots, u_m]$ for which $rank(B) \leq k$ and assume further that the $D_k - D_{k-1}$ is smooth. Let R denote $U\mathfrak{g}/(w)$.

Theorem 5.6. *Let \mathcal{B} be a pure Sullivan algebra, whose Lie model \mathfrak{g} is formal and as above. Let w be a potential which intersects the varieties D_k transversally at every point. Then:*

- (a) *w has isolated singularities*
- (b) *$\text{Proj}(R)$ has finite homological dimension as an abelian category.*

The first statement is a calculation, so we explain the second one. Consider the exact functor between derived categories

$$\pi : D^b(Gr - R) \rightarrow D^b(\text{Proj}(R))$$

We can consider the abelian subcategory of $Gr - R$, denoted $Gr - R_{\geq i}$ which consists of modules M such that $M_p = 0$ for $p \leq i$. Restricted to this subcategory,

$$\pi_{\geq i} : D_{\geq i}^b(Gr - R) \rightarrow D^b(\text{Proj}(R))$$

has a right adjoint

$$R\omega : D^b(\text{Proj}(R)) \rightarrow D_{\geq i}^b(Gr - R)$$

Thus we will show that for any $M, N \in D^b(Gr - R)$, $\text{Ext}^i(M, R\omega \circ \pi(N))$ vanishes for large i .

Suppose that Q is a graded prime ideal different from the maximal ideal and lying in a component of D_k , but not D_{k-1} . We denote $R/\text{rad}(QR)$ by B . Now denote by P the prime ideal corresponding to the irreducible component of D_k which Q is in. One can prove that the correspondence $P \mapsto \text{rad}(PR)$ gives a bijection between (graded) prime ideals in $\mathbb{C}[u_1, \dots, u_m]$ and (graded) prime ideals of $U\mathfrak{g}$ [Mus]. We have a short exact sequence:

$$0 \rightarrow S \rightarrow R/(\text{rad}(PR), Q) \rightarrow B \rightarrow 0$$

where S is B torsion by the assumption that the prime Q lie in a component of D_k but not D_{k-1} . Now we know by our condition, that $\mathbb{C}[u_1, \dots, u_m]/Q[l]$ has a finite resolution as a $\mathbb{C}[u_1, \dots, u_m]/P$ module and thus so does $R/(\text{rad}(PR), Q)[l]$ as a $R/\text{rad}(PR)$ module.

The above exact sequence reveals that $\text{Ext}_{R/(\text{rad}(PR))}^i(B[l], M)$ is B torsion for $i > m$. It is also easy to show from the transversality hypothesis that $R/\text{rad}(PR)[l]$ has finite homological dimension over R . Next, we note the following lemma, which is proved for ungraded rings in [Bro]

Lemma 5.7. *Let R be a graded FBN ring. Given a bounded complex C in $D(Gr - R)$ if $\text{Ext}^i(R/P[l], C)$ is R/P torsion for $i \gg j$ for every two-sided prime ideal P then $\text{Ext}^i(M, C)$ vanishes for $i \gg 0$.*

To finish the argument, we use a change of ring spectral sequence. Namely, we have a spectral sequence:

$$E^{pq} = \text{Ext}_{R/(\text{rad}(PR))}^p(R/\text{rad}(QR), \text{Ext}_R^q(R/\text{rad}(PR), M))$$

By the above discussion, E^{pq} is $R/\text{rad}(QR)$ torsion for $p > m$. Because $R/\text{rad}(PR)$ has finite homological dimension over R , E^{pq} vanishes for q sufficiently high, depending only on P . Therefore for large enough i only depending on P , $\text{Ext}_R^i(R/\text{rad}(QR), M)$ is torsion. Since there are only finitely many P that arise, the result follows from the previous lemma.

6. COMMENTS ON THE PURE SULLIVAN CONDITION

The condition that our dg-algebra be pure Sullivan may seem like a restrictive condition. To get a better feeling for why this is a natural condition if we are to expect a full open-closed field theory, we look at two examples, one where the rational homotopy type is hyperbolic and one where it is elliptic, but not pure Sullivan.

Example 6.1. Suppose now \mathcal{Q} is $(S^3 \times S^3 \times S^3) \# (S^3 \times S^3 \times S^3)$. A standard calculation in rational homotopy theory proceeds as follows:

Let N be the wedge $(S^3 \times S^3 \times S^3) \vee (S^3 \times S^3 \times S^3)$.

Then it is clear that

$$\pi_*(\Omega N) \otimes \mathbb{Q} \cong \text{Ab}(x_1, x_2, x_3) * \text{Ab}(x_4, x_5, x_6)$$

In this formula, $\text{Ab}(x_i, x_j, x_k)$ denotes the abelian Lie algebra generated by three even variables and $*$ denotes the free product of Lie algebras. Next consider the manifold given by $\mathcal{U} = S^3 \times S^3 \times S^3 - D$, where D is a small open disc in $S^3 \times S^3 \times S^3$.

$$\pi_*(\Omega \mathcal{U}) \otimes \mathbb{Q} \cong \text{Ab}(x_1, x_2, x_3) * \text{Free}(x)$$

Here $\text{Free}(x)$ denotes the free Lie algebra on one generator and $\deg(x) = 7$, which corresponds to the Whitehead triple product of three dimensional spheres. Next we have the following general formula in [FelHalTho], Theorem 24.7, for the rational homotopy Lie algebra of the connected sum of two manifolds M, N .

$$\pi_*(\Omega(M \# N)) \otimes \mathbb{Q} \cong \pi(\Omega M') * \pi(\Omega N') / (\alpha + \beta)$$

Here M' and N' are M and N with small discs removed and α and β are the attaching maps for the top cell.

In the case under consideration, the top cell is attached along the Whitehead product. This leads to the following calculation of homotopy groups for \mathcal{Q} :

$$\pi_*(\Omega \mathcal{Q}) \otimes \mathbb{Q} \cong \text{Ab}(x_1, x_2, x_3) * \text{Ab}(x_4, x_5, x_6) * \text{Free}(x)$$

The center of the universal enveloping algebra can be seen to be \mathbb{C} because the radical $R(\mathfrak{g})$ of the above Lie algebra is zero.

Lemma 6.2. *For any graded Lie algebra \mathfrak{g} in characteristic zero, such that each graded piece $\dim(\mathfrak{g}_i) < \infty$, there is a containment $Z(U\mathfrak{g}) \subset U(R(\mathfrak{g}))$*

Thus even on the homological level, the center of $H_*(\Omega\mathcal{Q})$ is given by \mathbb{C} . In view of the fact that the image of the map

$$HH^*(C_*(\Omega\mathcal{Q})) \rightarrow H_*(\Omega\mathcal{Q})$$

is contained in the center, there is no possibility for non-trivial curved deformations.

Even when the algebra is rationally elliptic and the image of the above morphism $HH^*(C_*(\Omega\mathcal{Q})) \rightarrow H_*(\Omega\mathcal{Q})$ is non-empty, there may be no compactifying deformation. Let \mathfrak{g} be a nilpotent finite dimensional lie algebra concentrated in even degree. Let \mathfrak{h} denote its center.

Lemma 6.3. *Suppose the natural morphism $Sym(\mathfrak{h}) \rightarrow Z(U\mathfrak{g})$ is surjective. Then for any potential, w , the curved category $(\widehat{\mathcal{A}}, w) - proj$ is either empty or non-compact.*

If w has any linear component, then one can compute that the Hochschild cohomology vanishes. If w is non-linear, let I denote the ideal $\mathfrak{g}U\mathfrak{g} \cap Sym(\mathfrak{h})$. Next, consider the curved module $M = \widehat{U\mathfrak{g}/I}$. We have that

$$Hom(M, M) \cong \widehat{U\mathfrak{g}/I} \otimes \Lambda(\mathfrak{h})$$

To see this let h_1, \dots, h_j denote a basis for \mathfrak{h} . We can write $w = \sum h_i w_i$. We let $(K_{(h_1, \dots, h_j)}(\widehat{U\mathfrak{g}}), d_0)$ denote the Koszul complex associated to the ideal I . Then the map $\wedge w_i dh_i$ defines a map:

$$d_1 : K_{(h_1, \dots, h_j)}^i(\widehat{U\mathfrak{g}}) \rightarrow K_{(h_1, \dots, h_j)}^{i+1}(\widehat{U\mathfrak{g}})$$

One can then see that $d_1 + d_0$ turns the $K_{(h_1, \dots, h_j)}(\widehat{U\mathfrak{g}})$ into a matrix factorization \mathcal{P} .

$$Hom(M, M) \cong Hom(\mathcal{P}, M) \cong \widehat{U\mathfrak{g}/I} \otimes \Lambda(\mathfrak{h})$$

We have $\dim_{\mathbb{C}}(Hom(M, M)) = \infty$ and thus the category is not compact.

Even if the map $Sym(\mathfrak{h}) \rightarrow Z(U\mathfrak{g})$ is not surjective, the above observation can be used to put strong restrictions on the possible curvings that can compactify the category. We do not pursue this further for reasons of space and interest.

Example 6.4. Let \mathfrak{g} be a nilpotent finite dimensional lie algebra of rank three, with product given by $[x_1, x_2] = x_3$ and all other brackets are zero. The universal enveloping algebra is the algebra $\mathcal{A} = \mathbb{C}\{x, y, z\}/(xy - yx = z)$. Here the center of the universal enveloping algebra is a polynomial ring $\mathbb{C}[z]$ and we consider curved modules (\mathcal{A}, z^n) . One can check that the category vanishes when $n = 1$. One could also try to deform using not only curved deformation of \mathcal{A} but also deform the higher multiplications. However, in this example, this does not affect the result.

Lemma 6.5. *There is no proper $\mathbb{Z}/2\mathbb{Z}$ -graded deformation of \mathcal{A} .*

Any such Maurer-Cartan solution would necessarily be of the form

$$p(z) + p_{12}(x, y, z)d/dx \wedge d/dy + p_{13}(x, y, z)d/dx \wedge d/dz + p_{23}(x, y, z)d/dy \wedge d/dz$$

where $p(z)$ is in $\mathbb{C}[z]((t))$ and $p_{ij}(x, y, z)$ are in $\mathbb{C}[x, y, z]((t))$. The fact that this satisfies the Maurer-Cartan equation implies that

$$p_{13}(x, y, z) = p_{23}(x, y, z) = 0$$

One can then compute that the $p_{12}(x, y, z)d/dx \wedge d/dy$ terms are exact and conclude that there are no proper deformations.

7. TANGENTIAL FUKAYA CATEGORIES

Given the close connection between string topology and the Floer theory of the cotangent bundle $T^*\mathcal{Q}$ explained in the introduction, we aim to give a Floer theoretic interpretation of our curved deformations of $C_*(\Omega\mathcal{Q})$. This construction was introduced independently by Nick Sheridan in his thesis [She] and the author in [Pom].

For motivation, let us consider the easiest case of a symplectic mirror to a Landau-Ginzburg model, that of S^2 . We think of a sphere as being the (open) disk bundle of the cotangent bundle, $D^*(S^1)$, compactified by the points at 0 and ∞ . This is then mirror to

$$(\mathbb{C}[z, z^{-1}], w = z + 1/z)$$

Work of [Sei3] proves that if we want to understand mirror symmetry for the Landau-Ginzburg model

$$(\mathbb{C}[z, z^{-1}], w = z^d + 1/z^d)$$

we can either consider the Fukaya category of the orbifold $S^2/(\mathbb{Z}/d\mathbb{Z})$, where $\mathbb{Z}/d\mathbb{Z}$ acts by rotations that fix the two points, or more concretely a Fukaya category where we require disks to intersect the compactifying divisor with ramification of order d .

This orbifold has a natural generalization. Consider a variety X and a collection of effective Cartier divisor \mathcal{D}_i , and d_i a collection of positive integers. The Cartier divisors define a natural morphism :

$$X \rightarrow [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

Definition 7.1. The *root stack* $X_{(D_i, d_i)}$ is defined to be the fibre product

$$X \times_{[\mathbb{A}^n/(\mathbb{C}^*)^n]} [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

where the map

$$[\mathbb{A}^n/(\mathbb{C}^*)^n] \rightarrow [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

is the d_i -power map.

There are three important properties of the root stack:

- (a) The root stack defines an orbifold, which has non-trivial orbifold stabilizers along the divisors
- (b) The coarse moduli space is exactly X and away from D_i the map $X_{(\mathcal{D}_i, d_i)} \rightarrow X$ is an isomorphism.
- (c) A map from a variety which is ramified to order d_i along the divisors \mathcal{D}_i lifts uniquely to a map to $X_{(\mathcal{D}_i, d_i)}$.

Let \mathcal{Q} be any simply connected manifold with a metric whose geodesic flow is periodic. There are three known families of examples, that of S^n ($n > 1$), $\mathbb{C}P^n$ and $\mathbb{H}P^n$. $T^*\mathcal{Q} - \mathcal{Q}$ then acquires a Hamiltonian S^1 action by rotating the geodesics (which then give rise to Reeb orbits when restricted to the unit cotangent bundle). This induces a natural Hamiltonian action on $(T^*\mathcal{Q} - \mathcal{Q}) \times \mathbb{C}$. The moment map for this Hamiltonian S^1 action

$$(T^*\mathcal{Q} - \mathcal{Q}) \times \mathbb{C} \rightarrow \mathbb{R}$$

is given by

$$(x, z) \mapsto H(x) + 1/2|z|^2$$

Where $H(x) = |x|$ is the Hamiltonian associated to the Hamiltonian action on $T^*\mathcal{Q} - \mathcal{Q}$. We then take the reduced space, that is the preimage of a regular value quotiented out by the S^1 action. Finally, we glue back in the zero section to obtain a manifold X which is a symplectic compactification of the open disk bundle $D^*(\mathcal{Q})$ by the smooth divisor \mathcal{D} .

When \mathcal{Q} is $\mathbb{C}P^n$, $X \cong \mathbb{C}P^n \times \mathbb{C}P^n$. Namely, we have an anti-holomorphic involution,

$$I : \mathbb{C}P^n \times \mathbb{C}P^n \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n$$

given by

$$(z, w) \rightarrow (\bar{w}, \bar{z})$$

Its fixed point set: $\mathcal{L} : \mathbb{C}P^n \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n$, is a Lagrangian submanifold, which corresponds to the zero section in the general construction. The divisor \mathcal{D} parameterizes oriented closed geodesics and is embedded as a $(1, 1)$ hypersurface, the locus where

$$\sum z_i w_i = 0$$

When \mathcal{Q} is S^n , we obtain the projective quadric Q_n and the divisor \mathcal{D} is the projective quadric Q_{n-1} . The zero section in the general construction corresponds to a vanishing sphere \mathcal{L} under a degeneration to a singular quadric.

Finally when $\mathcal{Q} \cong \mathbb{H}P^n$, we have that $X \cong Gr(2, 2n+2)$, the Grassmanian variety of two-planes in \mathbb{C}^{2n+2} [Akh].

In each of the cases we have

$$\pi_2(X, \mathcal{L}) \cong \mathbb{Z}$$

We want to consider three different moduli spaces of holomorphic disks.

Definition 7.2. We define the moduli space of tangential disks to be $\mathfrak{M}_{j,\ell}^d(X, \mathcal{L})$ the moduli space of holomorphic maps with the following extra data:

- (a) a map $u : (\mathbb{D}^2, S^1) \rightarrow (X, \mathcal{L})$
- (b) a collection of j points on the boundary
- (c) $u^{-1}(\mathcal{D}) = d(p_1 + p_2 + \cdots p_\ell)$ where p_j are points in $\mathbf{int}(\mathbb{D}^2)$

Definition 7.3. We define the auxiliary moduli space $MA_j^d(X, \mathcal{L})$ which parameterizes objects which consist of the following three pieces of data:

- (a) a collection of j points q_1, \dots, q_j on the boundary of a disk
- (b) two points p_1, p_2 in $\mathbf{int}(\mathbb{D}^2)$ such that there is a biholomorphism

$$\mathbb{D}^2 \rightarrow \mathbb{D}^2$$

which sends

$$p_1 \rightarrow -r, \quad p_2 \rightarrow r, \quad q_1 \rightarrow i$$

- (c) a map $(\mathbb{D}^2, S^1) \rightarrow (X, \mathcal{L})$ such that $u^{-1}(\mathcal{D}) = p_1 + (d-1)p_2$

Definition 7.4. The Mickey Mouse moduli space $MM_{j,\ell}^d(X, \mathcal{L})$ parameterizes objects which consist of the following three pieces of data:

- (a) A map from a nodal disk (u_1, u_2, u_3) with three components glued along marked points.
- (b) a collection of marked points on the boundaries of each of the disks
- (c) interior marked points p_1 and p_3 in u_1 and u_3 which intersect \mathcal{D} with multiplicities $d-1$ and 1 .

See Sheridan's thesis [She] for beautiful pictures of the above moduli spaces.

Using the moduli spaces of tangential disks, we now define a version of Floer theory for the Lagrangian submanifold $\mathcal{L} \subset X_{(\mathcal{D}, d)}$. We take the point of view that we will count in our theory holomorphic disks which intersect the boundary divisor with multiplicity d . As in relative Gromov-Witten theory, it is easy to achieve transversality for configurations of stable tangential disks, none of the components of which lie completely in the divisor. This is due to the following lemma in [CieMoh] which allows us to do all perturbations in a suitable open neighborhood V of our Lagrangian \mathcal{L} .

Lemma 7.5. *There is a Baire dense set of tamed almost complex structures $\mathcal{J}^{reg}(V)$ that agree with \mathcal{J}_0 outside V such that $\mathfrak{M}_{j,\ell}^d(X, \mathcal{L})$ is regular.*

The difficulty in defining such a theory in general is that under Gromov compactness, holomorphic curves may have components consisting of holomorphic spheres which live entirely in the divisor \mathcal{D} and the moduli space of such objects can often be non-regular since we cannot deform the complex structure in a neighborhood of the divisor. In our situation, this problem is more manageable because such configurations have high codimension in the moduli space of J -holomorphic spheres. First, recall that a symplectic manifold X is monotone if $\omega(X) = \tau c_1(TX)$ for $\tau \geq 0$. Because both \mathcal{D} and \mathcal{X} are monotone there is no problem with multiply covered curves. Next, we have the following lemma:

Lemma 7.6. *The moduli space of simple tangential J -holomorphic spheres is generically a pseudo-manifold. The configurations of tangential J -holomorphic spheres consisting of non-constant components living inside the divisor is codimension at least 2.*

We give the argument for $\mathcal{Q} = \mathbb{C}P^n$. The reader is encouraged to verify that the same argument is easily adapted to the case $\mathcal{Q} = S^n$ or $\mathbb{H}P^n$. The real dimension of the moduli-space of tangential J -holomorphic spheres with l intersections with the divisor is:

$$2n + 2c_1(u) - 6 - 2l(d - 1) = 2n + 2n(ld) - 6 + 2l$$

We must analyze what happens to tangential spheres under Gromov compactness. Let Y be the limit of a sequence of tangential spheres. Let Z be a component contained containing marked points z_1, \dots, z_m , and let α_i be the points in Z which glue to components that intersect the divisor at isolated points with multiplicity m_i . Y has the property that:

$$D \cdot Z + \sum_i m_i = dm$$

Because maps into the divisor \mathcal{D} are transverse, using the above dimension formula and the usual counting arguments for stable configurations of spheres, it is easy to verify that the codimension of such a configuration is $2 * (k - 1)$, where k is the number of components of the tree.

It is interesting to note that in the case $\mathcal{Q} = \mathbb{C}P^1$, we have the following calculation:

Lemma 7.7. $\mathfrak{M}_{j,\ell}^d(X, \mathcal{L})$ has the structure of an oriented pseudo manifold.

We make a transversality calculation for the open part of the moduli space in the case $\mathcal{Q} = \mathbb{C}P^1$. One can make a similar calculation for the various boundary components which arise under Gromov compactness. For this it is easier to work with the root stack $X_{(\mathcal{D},d)}$. We need to prove that the complex structure J is regular for disks in $X_{(\mathcal{D},d)}$. We have a map from $\pi : X_{(\mathcal{D},d)} \rightarrow X$, which gives rise to an exact sequence

$$0 \rightarrow TX_{(\mathcal{D},d)} \rightarrow \pi^*TX \rightarrow R \rightarrow 0$$

For any map $f : \mathbb{D}^2 \rightarrow X_{(\mathcal{D},d)}$, we get a sequence of sheaves on \mathbb{D}^2 , and using the reflection principle, we can double this to a sequence of sheaves on $\mathbb{C}P^1$. On $\mathbb{C}P^1$, we know that R is a skyscraper sheaf

of rank $2(d-1)\ell$, concentrated at the intersection point with \mathcal{D} and its opposite and that the double of the bundle f^*TX is of the form $\mathcal{O}(2d\ell) \oplus \mathcal{O}(2d\ell)$. We conclude that the double of $TX_{(\mathcal{D},d)}$ is also positive to deduce the desired result.

The author does not know whether this persists for S^{n+1} or $\mathbb{C}P^n$ for $n > 1$, though as noted above this is somewhat tangential to our main line of inquiry.

In any case, the theory can be defined along standard lines in two equivalent ways, either following [Sei2] or [FOOO].

Here we take the Morse Bott definition given in [FOOO] and we consider some model for chains, $C_*(X)((t))$, and using the evaluation maps

$$ev_i : \mathfrak{M}_{k+1,\ell}^d(X, \mathcal{L}) \rightarrow \mathcal{L}$$

to define a sequence of higher products

$$m_k(\alpha_1, \dots, \alpha_k) = \sum_{\ell} ev_{0,*}(\prod ev_i^*(\alpha_i))t^{\ell}$$

In our case, we are doing something slightly non-standard to our category by giving the Novikov-variable t a grading in order to relate it to the deformations we considered previously. This is valid here because the Lagrangian \mathcal{L} is a *monotone* Lagrangian submanifold. Recall that a Lagrangian \mathcal{L} is monotone if the two maps, corresponding to the action and Maslov index respectively:

$$A : \pi_2(X, \mathcal{L}) \rightarrow \mathbb{R}, \quad I : \pi_2(X, \mathcal{L}) \rightarrow \mathbb{Z}$$

satisfy the equation:

$$2A(u) = \tau I(u) \quad \forall u \in \pi_2(X, \mathcal{L})$$

Using the perturbation lemma, we have defined an A_{∞} deformation:

$$m_n : H^*(\mathcal{L})^{\otimes n}[[t]] \rightarrow H^*(\mathcal{L})[[t]]$$

the moduli spaces $\mathfrak{M}_{j,1}^d(X, \mathcal{L})$ in particular define classes e_d in $HH^*(C^*(\mathcal{Q}), C^*(\mathcal{Q}))$

Lemma 7.8. *The class e_d in $\mathfrak{M}_{j,1}^d(X, \mathcal{L})$ is gauge equivalent to the d -fold cup product e_1^d of the class defined by $\mathfrak{M}_{j,1}^1(X, \mathcal{L})$*

To prove this result for all d , we proceed by induction and consider the auxiliary moduli space. By standard Gromov compactness arguments, the boundary of the auxiliary moduli space consists of points where:

- (a) $r \rightarrow 0$, the boundary is the moduli space $\mathfrak{M}_{j,1}^d(X, \mathcal{L})$.
- (b) $r \rightarrow 1$ the boundary is the Mickey Mouse moduli space $MM^d(X, \mathcal{L})$.

The boundary as $r \rightarrow 1$ represents the Hochschild cup product of $e_{d-1} * e_1$

The boundary as $r \rightarrow 0$ is the class e_d .

This cobordism thus gives rise to the equation:

$$e_{d-1} * e_1 - e_d = \partial(MA)$$

where MA is the Hochschild cochain defined by the auxillary moduli space. This equation implies the result by induction.

It is interesting to note that the above proof follows the same line of reasoning as the proof in [FOOO] that bulk deformation :

$$H^*(X) \rightarrow \mathbf{HH}^*(Fuk(X))$$

is a ring homomorphism.

When $d = 1$ it is easy to calculate the relevant Hochschild cohomology class e_1 . In all cases, the cohomology ring is monogenerated. Namely we have that

$$C^*(\mathcal{L}) \cong \mathbb{C}[x]/(x^j)$$

Theorem 7.9. *The class e_1 is that which corresponds to the deformation*

$$HF(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}((t))[x]/(x^j - t)$$

For $\mathcal{Q} = \mathbb{C}P^n$ this follows from the fact that $HF^*(\mathcal{L}, \mathcal{L}) \cong QH^*(\mathbb{C}P^n)$, which is known to agree with the above ring [FOOO]. For $\mathcal{Q} = S^n$, this result is contained in [Smi]. We give a brief synopsis. To every *weakly unobstructed* Lagrangian in $Fuk(X)$, we assign a number m_0 [FOOO]. Let $Fuk(X, 0)$ denote the subcategory generated by Lagrangians such that $m_0 = 0$. Smith shows that the Lagrangian \mathcal{L} generates this subcategory. Let $QH^*(X, 0)$ denote the 0-eigenspace under the map.

$$c_1 \cup : QH^*(X) \rightarrow QH^*(X)$$

Smith proves that $QH^*(X, 0) \cong HH^*(Fuk(X, 0))$. It follows by Maslov index considerations that as a vector space $HF^*(\mathcal{L}, \mathcal{L}) \cong H^*(S^n)$. Beauville [Bea] calculated the quantum cohomology of X . The only A_∞ structure (up to gauge equivalence) on $H^*(S^n)$ which gives the correct HH^* is the one above.

Finally, we verify this if $\mathcal{Q} = \mathbb{H}P^n$. We have the following lemma for the Grassmannian $X = Gr(2, 2n+2)$:

Lemma 7.10. $QH^*(X, 0) \cong \mathbb{C}((t))[x]/(x^j - t)$

To perform this calculation, we first note that there is the following presentation of the cohomology of the Grassmannian X . Let x_1 and x_2 denote the Chern classes of the tautological bundle E and y_1, \dots, y_{2n+2} denote the Chern classes of the complementary bundle F . The fact that $E \oplus F$ is topologically trivial implies the following relation among the Chern classes for any positive integer j

$$\sum_i x_i y_{j-i} = 0$$

This relation combined with the vanishing of y_{2n+1} and y_{2n+2} gives the following presentation for the cohomology of the Grassmannian:

$$H^*(Gr(2, 2n+2)) \cong \mathbb{C}[x_1, x_2]/(y_{2n+1}, y_{2n+2})$$

The quantum deformation is known to be given by $QH^*(X) \cong \mathbb{C}[x_1, x_2, t]/(y_{2n+1}, y_{2n+2} - t)$. Focusing on the relation:

$$y_{2n+2} = x_2 * y_{2n} + x_1 * y_{2n+1}$$

we observe that upon setting $x_1 = 0$, which corresponds to taking the 0-eigenspace of multiplication by the first Chern class, $y_{2n+2} = x_2^{n+1}$. This concludes the proof.

To finish the proof of the theorem we note that the gradings of $QH^*(X, 0)$ and $HF^*(L, L)$ are well defined modulo $4n+4$ and the map $QH^*(X, 0) \rightarrow HF^*(L, L)$ must preserve the gradings. Hence we conclude that this map is both injective and surjective since each element $1, x, x^2, \dots, x^n$ in $QH^*(X, 0)$ is invertible.

Returning to our main calculation, we have a “finite determinacy” lemma. We state it for $\mathbb{C}P^n$, but the obvious adaptation of the theorem to the cases where $\mathcal{Q} = S^n$ or $\mathbb{H}P^n$ also holds.

We first explain what we know about the $A(\infty)$ structure on $HF_{X_{D,d}}^*(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}[e]/e^{n+1}((t))$. We can write

$$m = tm_{2d} + \sum t^k \tilde{m}^k$$

where $k \geq 2$ and the arity of \tilde{m}^k is even, larger than $2d$, and increasing with k . Note that for $d = 1$ the above isomorphism is not an algebra map as the multiplication m_2 is deformed.

We have the following formula for m_{2d}

$$m_{2d}(e^{a_1}, e^{a_2}, \dots, e^{a_{2d}}) = t, \quad \text{if} \quad \sum (a_i) = (n+1)d$$

Lemma 7.11. *The $A(\infty)$ structure on $HF_{X_{D,d}}^*(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}[e]/e^{n+1}((t))$ is determined by the fact that $m_j = 0$, $2 < j < 2d$ and $m_{2d}(e^{a_1}, e^{a_2}, \dots, e^{a_{2d}}) = t$, if $\sum (a_i) = (n+1)d$*

We use the model for Hochschild cohomology developed in section 4 namely we imagine a Maurer-Cartan solution inside of

$$T^{poly}(C[x] \otimes \Lambda[\beta])[[t]]$$

of the form

$$x^n d/de + (d/de^n)t + \sum_k \tilde{m}^k t^k$$

For the lowest k appearing in the sum above, it follows by a calculation of the Gerstenhaber algebra structure on $HH^*(C^*(\mathbb{C}P^n), C^*(\mathbb{C}P^n))$ that $[\tilde{m}^k, x^n d/de + (d/de^n)t] = 0$.

By calculations similar to those presented at the beginning of this section, \tilde{m}^k is necessarily exact and thus there is a class p such that $b(p) = \tilde{m}^k$. One can thus write down an $A(\infty)$ change of coordinates given by the formula $id + p$ which eliminates \tilde{m}^k and one can see that this only effects products of higher degree. Continuing in this way, one can prove the desired lemma.

By a Kunneth theorem, we can get similar results for manifolds of the form $\mathcal{Q} = \prod \mathbb{C}P^{n_j}$.

8. CONNECTIONS TO SYMPLECTIC FIELD THEORY

One can give a uniform treatment of these results in the context of symplectic field theory as well. Seidel [Sei] has defined open-closed string maps from

$$S : SH^*(D^*(\mathcal{Q}), b) \rightarrow HH^*(Fuk(D^*(\mathcal{Q}), b))$$

$$S^{eq} : SH_{eq}^*(D^*(\mathcal{Q}), b) \rightarrow CC^*(Fuk(D^*(\mathcal{Q}), b))$$

This map is defined below. We only work with respect to the background class induced by $w_2(\mathcal{Q})$ and so drop this from the notation. Denote $D^*(\mathcal{Q})$ by W and by \widehat{W} its symplectic completion, which is $T^*\mathcal{Q}$. We consider a class of Hamiltonians

$$H : \widehat{W} \rightarrow \mathbb{R}$$

such that

- (a) on W , the function H is Morse and taken to be \mathcal{C}^2 small, and satisfy $H \leq 0$ in this region
- (b) in the region $S^*(M) \times \mathbb{R}^+$, $H(x, r)$ is a function $h(e^r)$, where h is a strictly increasing function, with $h = \alpha e^{2r} + \beta_0$ for $r > r_0$, $\alpha, \beta_0, r_0 \in \mathbb{R}^+$

Let $S(H)$ denote the set of time one periodic flows σ of the Hamiltonian H . To make things explicit, in the region $S^*(\mathcal{Q}) \times \mathbb{R}^+$, X_H has the form $h'(e^r)R$, where R is the Reeb vector field and time one orbits of X_H can be identified with orbits of the Reeb vector field of time $h'(e^r)$.

We use the following definition of symplectic cohomology [BouEkhElia]. Throughout this section we use the conventions in this paper for gradings and Morse homology. Assuming $S(H)$ is discrete, let CH denote the vector space generated by $S(H)$. The complex for symplectic cohomology contains

two distinct copies of CH^\vee and CH^\wedge which come from the fact that because our Hamiltonian is time-independent each Reeb orbit gives rise to an S^1 -family of 1-periodic orbits. For any Reeb orbit σ , we denote this moduli space by S_σ .

Picking a generic Morse-function for each S_σ with exactly two critical points, the generators CH^\vee and CH^\wedge correspond to the maximum M and minimum m of this Morse function.

In loose terms, the differential in $SH^*(W)$ is given by counting elements of the moduli space $M(S_{\sigma_+}, S_{\sigma_-})$ of cylinders which solve Floer's equation:

$$u : \mathbb{R} \times S^1 \rightarrow \hat{W}$$

- (a) $\bar{\partial}u = \mathcal{J}(X_H)$
- (b) $\lim_{r \rightarrow \infty} u(r, \theta) \rightarrow \alpha \subset S_{\sigma_+}$
- (c) $\lim_{r \rightarrow -\infty} u(r, \theta) \rightarrow \beta \subset S_{\sigma_-}$

To be precise, we notice that given any two Reeb orbits σ_+ , σ_- , we have canonical evaluation maps

$$ev_+ : M(S_{\sigma_+}, S_{\sigma_-}) \rightarrow S_{\sigma_+}$$

$$ev_- : M(S_{\sigma_+}, S_{\sigma_-}) \rightarrow S_{\sigma_-}$$

The differential

$$d(\sigma_{+, \wedge}) = \Sigma_{\sigma_-} N(\sigma_{+, \wedge}, \sigma_{-, \wedge}) \sigma_{-, \wedge} + N(\sigma_{+, \wedge}, \sigma_{-, \vee}) \sigma_{-, \vee}$$

Where $N(\sigma_{+, \wedge}, \sigma_{-, \vee/\wedge})$ counts curves u such that $ev_+(u)$ lies in the unstable locus of m and $ev_-(u)$ lies in the stable locus of M/m . The obvious variation of the above formula holds for $\sigma_{+, M}$.

The definition of SH_{eq}^* is similar except that the complex is generated by a single copy of $S(H)$, and the differential simply counts Floer trajectories between Reeb orbits.

In this situation, Seidel defined the above maps by first picking a radially symmetric function $p(r) : \mathbb{D}^2 \rightarrow \mathbb{R}$, which is 0 in a neighborhood of the boundary and 1 in a neighborhood of the origin. Denote by H_r the function $p(r)H$.

For any Lagrangian \mathcal{L} in W , we consider Floer interpolation trajectories, that is maps

$$u : \mathbb{D}^2 - \{0\} \rightarrow \widehat{W}$$

which satisfy

- (a) a deformed Floer equation $\bar{\partial}u = \mathcal{J}(X_{H_r})$
- (b) $\lim_{r \rightarrow 0} u(r, \theta) \rightarrow \alpha$
- (c) $u(\partial\mathbb{D}^2) \in \mathcal{L}$

Seidel's maps are then defined by the obvious generalization of the ordinary A_∞ product in Lagrangian Floer theory using the moduli space of such trajectories. See the above reference for a schematic drawing.

To reveal the relation between the tangential Fukaya category and Seidel's map, we would like to modify the definition of symplectic cohomology in two distinct ways. Using compactness results of [BouEkhElia] allows one to take the limit as $H \rightarrow 0$ and define symplectic cohomology purely in terms of holomorphic curves. Additionally, in our case, the space of Hamiltonian orbits is not discrete, but one can define appropriate Morse-Bott versions of $SH^*(D^*(\mathcal{Q}))$, $SH_{eq}^*(D^*(\mathcal{Q}))$ and S^{eq} [BouEkhElia].

In our case, the result is as follows. We begin by selecting a generic perfect Morse function on \mathcal{D} . The vector space CH is given by a copy of $H(\mathcal{D})$ for each positive natural number, which correspond to the space of Reeb orbits and their multiple covers. Symplectic cohomology is given by the following complex.

$$CH^\vee \oplus CH^\wedge[1] \oplus Morse(-H)[-n], d$$

where d is described by the following matrix .

$$d = \begin{vmatrix} 0 & 0 & 0 \\ d_0 & 0 & 0 \\ d_1 & 0 & d_{Morse} \end{vmatrix}.$$

For $(\alpha, d)^\vee$ and $(\beta, d')^\wedge$ we consider the moduli space $M((\alpha, d), (\beta, d'))$ of holomorphic cylinders with asymptotic markers asymptotic at $+\infty$ to a Reeb orbit of multiplicity d in the unstable locus of α and asymptotic to a Reeb orbit of multiplicity d' in the stable locus of β .

$$d_0((\alpha, d)^\vee) = (c_1(N(D)) \cap (\alpha, d)^\wedge + \Sigma_{(\beta, d')} |M((\alpha, d), (\beta, d'))| (\beta, d')^\wedge$$

For any class α, d and critical point p of H , we define $M(\alpha, p)$ to be the moduli space of holomorphic maps

$$u : \mathbb{C} \rightarrow T^*(\mathcal{Q})$$

such that $u(0)$ is contained in the stable manifold of p and u is asymptotic to a Reeb orbit of multiplicity d in the unstable locus of α . When this moduli space is zero dimensional we say that

$$d_1(\alpha) = \Sigma_p |M(\alpha, p)| p$$

In this setting, Seidel's map is defined by the analogous count of holomorphic curves with Lagrangian boundary conditions and asymptotic conditions at Reeb orbits.

We are now in a position to relate the tangential Fukaya category to SFT. Namely, by a removal of singularities argument, punctured \mathcal{J} -holomorphic half cylinders asymptotic to d -fold covers of Reeb orbits in $T^*(\mathcal{Q})$ are in bijection with maps $\mathbb{D}^2 \rightarrow X$ tangent to \mathcal{D} with tangency of order d . Thus, the infinitesimal deformations defined above in the definition of the d -fold tangential

Fukaya category correspond to the d -th copy of the fundamental class of \mathcal{D} inside of CH^\wedge under Seidel's map.

Remark 8.1. More generally, in the formalism above, if \mathcal{X} is a projective variety and \mathcal{D} is a smooth ample divisor, one could pick a generic Morse function on \mathcal{D} and consider the unstable manifold of critical points p_i in \mathcal{D} which represent classes of $H_*(\mathcal{D})$. One can then define an infinitesimal deformation by considering as above disks with a single point of intersection with \mathcal{D} of multiplicity d , but additionally requiring that the point of tangency simultaneously lie in W_{p_i} . Based upon the analysis using the auxillary moduli space in the previous section, we expect that the quantum multiplication of these classes is governed by the quantum product in \mathcal{D} , though we have not checked the details as these classes play a minor role in our story.

Remark 8.2. Under the above open closed string map, CH^\vee correspond to considering disks with tangency conditions and an asymptotic marker at the intersection point with the divisor. These cycles are used in ongoing work by Abouzaid and Smith in their study of Khovanov Homology. Note that they define Hochschild cohomology classes precisely when the above differential vanishes. In his thesis, Luís [Diogo] develops the above picture of symplectic homology much further by expressing all of the various differentials in terms of Gromov-Witten invariants for \mathcal{D} and \mathcal{X} .

9. CONNECTIONS TO HOMOLOGICAL MIRROR SYMMETRY

The above analysis of the tangential Fukaya category gives a mirror construction for a wide variety of Fano varieties. Our discussion in this section overlaps greatly with work of [Gross, Keel, and Hacking], which is motivated by tropical considerations. Let X be a projective variety and $D = \cup D_i$ an ample strict normal crossings divisor such that $c_1(X - D) = 0$. We denote the complement $X - D$ by U .

To each component of the intersection, $D_{i_0} \cap D_{i_2} \cap \dots \cap D_{i_p}$, we would like to define a Hochschild class of $Fuk(X - D)$, $\alpha_{([i_0, i_1, \dots, i_p], j)}$, which counts holomorphic disks with Lagrangian boundary and which intersect the j -th component of the intersection $D_{i_0} \cap D_{i_2} \cap \dots \cap D_{i_p}$ simply and intersect none of the other components. As we saw above, technical points need to be addressed to define these classes in general. A future paper will explain how to overcome these technical issues and discuss extensions of the classes to the wrapped Fukaya category. We would like to propose the following picture in this situation.

Conjecture 9.1. *Let A be the subalgebra of $HH^*(Fuk(U))$ generated by the classes $\alpha_{I,j}$ above. For any object in the wrapped Fukaya category, L , the natural mapping $A \rightarrow HF(L, L)$ is module finite. Let X be a Fano variety and D an anti-canonical divisor, then the sub-algebra A of Hochschild cohomology generated by the $\alpha_{I,j}$ coincide with global sections $\Gamma(\mathcal{O}_{U^\vee})$. The mirror of X is given by the LG-model $(U^\vee, \alpha_0 + \alpha_1 + \dots + \alpha_n)$*

The most extreme case of the conjecture is consistent with that proposed in [Aur], D is a smooth anti-canonical divisor and the mirror, U^\vee admits a proper map

$$\alpha : U^\vee \rightarrow \mathbb{A}^1$$

the general fiber of which is mirror to D . In this setting $\Gamma(O_U^\vee) = \mathbb{C}[\alpha]$ and elements of the form α^d correspond to the infinitesimal deformation underlying the tangential Fukaya category.

This conjecture is also motivated by the calculations in section 6. We first note the following strong result due to McLean from a recent paper [McL].

Theorem 9.2. *If $T^*\mathcal{Q}$ is symplectomorphic to an affine variety A , then \mathcal{Q} is (rationally) elliptic.*

Rephrasing, if there is a projective symplectic manifold X , and an ample normal crossings divisor $\cup \mathcal{D}_i$, such that $X - \cup \mathcal{D}_i \cong T^*\mathcal{Q}$, then \mathcal{Q} is rationally elliptic. McLean proves this result by proving that under the existence of such a compactification, $\text{rank}(SH^k(A))$ has polynomial growth in k . The degree of the growth is bounded by the maximal number of components which have non-empty intersection.

On the other hand, the author does not know of an example of such a cotangent bundle for which \mathcal{Q} is not Pure Sullivan. The above conjecture, if true, would imply that McLean's result could be further strengthened, and in particular that the examples of section 6 could not arise as affine varieties.

Now suppose that X is Fano and D has at least one intersection of maximal codimension. We define the naive mirror of $X - D$, which we will denote by U^\vee , to be a crepant resolution of $\text{Spec}(A)$, should such a crepant resolution exist. We now show in several examples that this construction generalizes several known mirror constructions. In all of the examples below, a similar analysis to that done in the previous section yields that these Hochschild classes are well defined. Consideration of the boundary of the auxillary moduli space will again yield the rules for multiplication. One component of the boundary will always correspond to the Hochschild multiplication. Unlike in the tangential Fukaya category above, sphere bubbles may arise which gives rise to an enumerative problem to compute the multiplication rules.

Example 9.3. $X = \mathbb{C}P^n$ and let D be the toric normal crossings divisor. Then for any holomorphic sphere which bubbles in the auxillary moduli space necessarily intersects all of the divisors. Thus, if I and J generate a proper subset of $0, 1, \dots, n$, we have that

$$\alpha_I \alpha_J = \alpha_{I+J}$$

Next we determine $\alpha_{0,1,2,\dots,n-1} * \alpha_n$. There is a single holomorphic sphere which passes through the intersection $D_0 \cap D_1 \cap \dots \cap D_{n-1}$ and any other marked point in projective space. This holomorphic sphere intersects the divisor D_n in a single point. Thus, in this case, the boundary of the auxillary moduli space consists of these holomorphic spheres glued to holomorphic disks in U . This gives rise to the equation

$$\alpha_{0,1,2,\dots,n-1}\alpha_n = q$$

Thus the variety U^\vee is given by the subvariety $\alpha_0\alpha_1\cdots\alpha_n = q$ inside of \mathbb{C}^{n+1} . The mirror to $\mathbb{C}P^n$ is then the usual toric mirror, namely $(U^\vee, \alpha_0 + \alpha_1 + \cdots + \alpha_n)$. A similar calculation should hold for all toric varieties.

It is interesting to consider what happens when we remove another generic hyperplane H giving rise to the pair of pants, a symplectic manifold which was spectacularly exploited by Sheridan in his proof of the homological mirror symmetry conjecture for Fano and Calabi-Yau projective hypersurfaces [?]. Now the configurations of sphere bubbles disappear because they intersect the hyperplane H , and the algebra A is given by the commutative algebra on $n+2$ generators where the product of $n+1$ generators vanish. $\text{Spec}(A)$ now corresponds to the singular locus of the mirror

$$(\mathbb{C}[x_0, x_1, \dots, x_{n+1}], x_0x_1\cdots x_{n+1})$$

This provides further evidence for the first statement of the above conjecture.

Example 9.4. The next example comes from [GroKeeHac], rephrased in the language of symplectic geometry. Let X be a degree 5 Fano surface and D a cyclic anticanonical divisor of -1 curves.

We first note that we must have

$$\alpha_i\alpha_{i+1} = \alpha_{i,i+1}$$

, where in this equation the divisors are ordered cyclically. This is because any bubbling configuration in the divisor would necessarily intersect more components than just D_i and D_{i+1} and an inspection of the Gromov Witten theory of such a Fano surface [CrauMir] shows that no other configuration is possible. The multiplications for

$$\alpha_i\alpha_{i+2}$$

are slightly more intricate. Here there are two possible configurations of holomorphic sphere bubbles. The first is given by a sphere which is mapped isomorphically to the divisor D_{i+1} glued to a holomorphic disk with Lagrangian boundary that intersects D_{i+1} at one point. The second is determined by a unique holomorphic sphere B which intersects a marked point on the divisor D_i and D_{i+1} and a given marked point in the complement of the divisor. Thus the equation:

$$\alpha_i\alpha_{i+2} = q^{\omega[B]} + \alpha_{i+1}$$

Example 9.5. The following calculation generalizes work of [Aur]. $X = \mathbb{C}P^n$ and D be the normal crossings divisor given by the union of D_0 , a conic, and $n-1$ -lines and D_1, \dots, D_{n-1} in general position. We construct the mirror for $X - D$. As above, it is easy to see that for any collection I and J which span a collection of length less than n

$$\alpha_I\alpha_J = \alpha_{I+J}$$

We have the following formula for the collection of length n ,

$$\alpha_{1,\dots,n-2}\alpha_0 = \alpha_{[0,\dots,n-1],0} + \alpha_{[0,\dots,n-1],1}$$

Finally, we have the equation

$$\alpha_{[0,\dots,n-1],0}\alpha_{[0,\dots,n-1],1} = q\alpha_{1,\dots,n-2}$$

given by the holomorphic sphere $D_1 \cap D_{n-1}$.

Example 9.6. This is an example which is more relevant to us. Consider the quadric X with D the anticanonical divisor given by the union of n -generic hyperplanes D_i . Clearly, if I and J generate a proper subset of the collection $0, 1, \dots, n-1$,

$$\alpha_I\alpha_J = \alpha_{I+J}$$

We have the following relations coming from the Gromov Witten theory of X .

$$\alpha_{n-1}\alpha_{0,1,\dots,n-2} = \alpha_{[0,1,\dots,n-2,n-1],0} + \alpha_{[0,1,\dots,n-2,n-1],1} + 2q$$

This variety is singular, but has a crepant resolution U^\vee , which is mirror to $X - D$. The mirror for the quadric is given $(U^\vee, \alpha_0 + \dots \alpha_{n-1})$. The mirror of T^*S^n is given by $(U^\vee, \alpha_0 + \dots \alpha_{n-2})$. One can also consider the same construction to obtain a mirror for $T^*\mathbb{C}P^n$. We will explore this mirror in the context of hyperkähler geometry in a future paper.

We now examine more closely the case of $\mathbb{C}P^1 \times \mathbb{C}P^1$. The reader will notice that the above results on the Fukaya category were not fully satisfactory as they described only the subcategory of $Fuk(\mathbb{C}P^1 \times \mathbb{C}P^1)$ generated by the zero-section \mathcal{L} as a curved deformation, rather than the full Fukaya category. In this section, we show that by regarding $C_*(\Omega S^2)$ as a $\mathbb{Z}/2\mathbb{Z}$ -graded dga, we may realize the full Fukaya category as a curved deformation.

Specializing the formula given in section 1.2 for $C_*(\Omega\mathbb{C}P^n)$ to when $n = 1$, we obtain that:

$$C_*(\Omega S^2) \cong \mathbb{C}[u] \otimes \Lambda(e)$$

with $m_2(e, e) = u$. Here we have that $|u| = 2$ and $|e| = 1$.

We examine in more detail the mirror to T^*S^2 . The mirror was constructed above, but it will be useful for what follows in the next two sections to understand its construction from an SYZ point of view. This construction is a special case of [?].

For ϵ a real constant, we view T^*S^2 as a hypersurface in \mathbb{C}^3 cut out by the equation:

$$Y_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : xy - z^2 = \epsilon\}.$$

We have an action of S^1 on this manifold given by

$$\alpha * (x, y, z) = (\alpha x, \alpha^{-1}y, z)$$

We also have a mapping $z : Y_\epsilon \rightarrow \mathbb{C}$ which defines a Lefschetz fibration. Picking a real number δ larger than $\sqrt{\epsilon}$ and we work relative to the divisor

$$D_\delta = \{(x, y, z) \in Y_\epsilon, z = \delta\}.$$

There is a special Lagrangian fibration defined on the complement of this divisor given by

$$L_{r, \tilde{\lambda}} = \{(x, y, z, w) \in Y_\epsilon : |z - \delta| = r, \mu(x, y, z) = \lambda\}.$$

Here $\mu(x, y, z)$ is the moment map for the above S^1 action. One can construct the SYZ mirror to this hypersurface and we obtain the following:

Let

$$X = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* : uv = (1 + w)^2\}$$

X has a canonical small resolution $\pi : \tilde{X} \rightarrow X$ given by considering the subvariety of $\mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}P^1$ cut out by the following equations:

$$\tilde{X} = \{(x, y, z, t_1, t_2) \in \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}P^1 : vt_1 = (1 + w)t_2, ut_2 = (1 + w)t_1\}$$

Theorem 9.7. *The mirror to Y_ϵ is given by the LG-model (\tilde{X}, v) .*

We consider two categories of matrix factorizations, which correspond to the \mathbb{Z} -graded and $\mathbb{Z}/2\mathbb{Z}$ -graded modules over $C_*(\Omega S^2)$ respectively. To construct the $\mathbb{Z}/2\mathbb{Z}$ -graded category of matrix factorizations in this non-affine setting, we now consider the dg category of curved complexes of coherent sheaves $\text{Coh}(\tilde{X}, v)$, that is, the category with objects:

$$E = (E_1 \begin{array}{c} \xrightarrow{e_1} \\ \xleftarrow{e_0} \end{array} E_0)$$

where the E_i are coherent sheaves of $\mathcal{O}_{\tilde{X}}$ -modules and the e_i are morphisms of $\mathcal{O}_{\tilde{X}}$ -modules satisfying $e_{i+1} \circ e_i = v \cdot \text{id}_{E_i}$. The morphism complexes are defined exactly as before except with $\text{Hom}_{\mathcal{O}_{\tilde{X}}}$ rather than Hom_A .

Definition 9.8. Denote by $\text{Acycl}^{\text{abs}}[\text{Coh}(\tilde{X}, v)] \subset [\text{Coh}(\tilde{X}, v)]$ the thick triangulated subcategory generated by the total curved complexes of exact triples of curved quasi-coherent \mathcal{O}_X -modules. Objects of $\text{Acycl}^{\text{abs}}[\text{Coh}(\tilde{X}, v)]$ are called *acyclic*. The triangulated category $\text{MF}(\tilde{X}, v)$ is defined to be the quotient triangulated category

$$[\text{Coh}(\tilde{X}, v)] / \text{Acycl}^{\text{abs}}[\text{Coh}(\tilde{X}, v)]$$

This definition is also used in [Pos, Or1].

There is an embedding $i : \mathbb{A}^1 \hookrightarrow \tilde{X}$ defined by restricting to the locus where $t_2 = 0$. This identifies with the singular locus of the function v .

It is easy to see using results of [LinPom] that the structure sheaf for \mathbb{A}^1 generates the category $MF(\tilde{X}, v)$. Furthermore, a calculation reveals that

$$Hom(\mathbb{A}^1, \mathbb{A}^1) \cong C_*(\Omega S^2)$$

Here is how that calculation goes. The singular locus is contained in the part where $t_1 \neq 0$. Therefore we can consider a new variable $q = t_2/t_1$. The neighborhood $t_1 \neq 0$ is an affine space with variables q, u . In these coordinates, the potential $v = q^2 u$. The object that we want to consider is the brane defined by $q = 0$. As it is no different, and this will be useful later, we consider this brane in the category of modules over $(\mathbb{C}[u, q], q^{p+1}u^p)$. Following the model category structure introduced in [LinPom], we can construct a matrix factorization which resolves this coherent matrix factorization as follows:

$$P = (\mathbb{C}[u, q] \begin{array}{c} \xrightarrow{q} \\ \xleftarrow{u^p q^p} \end{array} \mathbb{C}[u, q])$$

From here we have that $Hom(P, P) \cong Hom(P, \mathbb{C}[u, q]/q)$. The differential vanishes on this latter complex and we have that

$$Hom(P, \mathbb{C}[u, q]/q) \cong \mathbb{C}[u] \otimes \Lambda(e)$$

Chasing through the isomorphism $Hom(P, P) \cong Hom(P, \mathbb{C}[u, q]/q)$, we get that $m_{p+1}(e, e, \dots, e) = u^p$ as claimed.

One point to remember is that we must view $C_*(\Omega S^2)$ as a $\mathbb{Z}/2\mathbb{Z}$ graded object.

In the above notation, now just considering the category of matrix factorizations over $(\mathbb{C}[u, q], q^2 u)$, it will be helpful to record the endomorphisms of the brane defined by $u = 0$. The corresponding matrix factorization is given by:

$$P' = (\mathbb{C}[u, q] \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{q^2} \end{array} \mathbb{C}[u, q])$$

Then we have that $Hom(P', P') \cong Hom(P', \mathbb{C}[u, q]/u) \cong \mathbb{C}[q]/q^2$

This corresponds to the fact that the exceptional $\mathbb{C}P^1$ is mirror in \tilde{X} to the zero section in T^*S^2 . Notice that there is \mathbb{C}^* action on \tilde{X} , given by,

$$\alpha * (u, v, w) = (\alpha^2 u, \alpha^{-2} v, w)$$

Notice that v has weight -2 with respect to this \mathbb{C}^* action.

We now describe how to use this \mathbb{C}^* - action to define a graded refinement of this category.

Definition 9.9. A *graded coherent D-brane* is an equivariant coherent sheaf E , equivariant with respect to a \mathbb{C}^* action, equipped with an endomorphism d_E of weight one, such that $d_E^2 = v$. The term *graded matrix factorization* is reserved for the case when E is a vector bundle.

The category of graded coherent D-branes has a triangulated structure given by shifting the weights. As usual, given two graded coherent D-branes E and F , the complex of morphisms

$$Hom(E, F), d \circ f - (-1)^{|f|} f \circ d$$

makes the category into a differential graded category.

We write the formula for the differential so that the reader can note that it has degree 1 and the category $[Coh_{\mathbb{C}^*}(\tilde{X}, v)]$ is \mathbb{Z} -graded triangulated category. As above, we can define the category $Acycl^{abs}[Coh_{\mathbb{C}^*}(\tilde{X}, v)]$ and we define $MF_{\mathbb{C}^*}(\tilde{X}, v)$ as above.

Lemma 9.10. *The structure sheaf \mathbb{A}^1 generates the category $MF_{\mathbb{C}^*}(\tilde{X}, v)$.*

We have an obvious functor

$$F : MF_{\mathbb{C}^*}(\tilde{X}, v) \rightarrow MF(\tilde{X}, v)$$

In [Shi] it is proven that the idempotent completion of the category $MF_{\mathbb{C}^*}(\tilde{X}, v)$ embeds as the compact objects into the absolute derived category of graded *quasicoherent* D-branes. Therefore, it is enough to show that if P is a graded matrix factorization resolving \mathbb{A}^1 for any quasi-coherent curved module \mathcal{J} , $Hom(P, \mathcal{J})$ is never zero [BonVDB]. Given the graded matrix factorization P and any graded quasi-coherent D-brane, \mathcal{J} , it is proven in [Shi] and [LinPom] that the space of morphisms $Hom(P, \mathcal{J})$ is simply a \mathbb{Z} -graded refinement of the space of morphisms of the space of morphisms between $F(P)$ and $F(\mathcal{J})$. If P is an equivariant matrix factorization resolving \mathbb{A}^1 , then the fact that $Hom(F(P), F(\mathcal{J}))$ is never zero, implies that $Hom(P, \mathcal{J})$ is never zero.

Notice that with this grading, the variable u has degree 2, which makes the isomorphism:

$$Hom(\mathbb{A}^1, \mathbb{A}^1) \cong C_*(\Omega S^2)$$

into a graded isomorphism.

Following Auroux's paper, one can determine the deformation which corresponds to further compactification of Y_ϵ to $\mathbb{C}P^1 \times \mathbb{C}P^1$. The standard Hori-Vafa mirror for $\mathbb{C}P^1 \times \mathbb{C}P^1$ is given by

$$((\mathbb{C}^*)^2, z_1 + z_2 + 1/z_1 + 1/z_2)$$

Our mirror is a partial compactification of the toric mirror, and there is a coordinate transformation given by

$$v = z_1 + z_2, u = (z_1 + z_2)/z_1^2, w = z_2/z_1$$

Therefore we conclude that:

Lemma 9.11. *Compactification corresponds to the deformation u/w .*

Notice that this is not a deformation of the category $MF_{\mathbb{C}^*}(\tilde{X}, v)$ only of the category $MF(\tilde{X}, v)$. We denote the $\mathbb{Z}/2\mathbb{Z}$ graded Hochschild cohomology by $HH_{\mathbb{Z}/2\mathbb{Z}}^*(C_*(\Omega S^2))$. The classes u and u/w are equivalent as Hochschild classes in $HH_{\mathbb{Z}/2\mathbb{Z}}^*(C_*(\Omega S^2))$, but not as Maurer-Cartan classes in $HH_{\mathbb{Z}/2\mathbb{Z}}^*(C_*(\Omega S^2)((t)))$.

We note that this claim does not contradict the formality lemma in the previous section. Note that $HH_{\mathbb{Z}/2\mathbb{Z}}^*(C_*(\Omega S^2)((t)))$ is *not* isomorphic to $HH_{\mathbb{Z}/2\mathbb{Z}}^*(C^*(S^2)((t)))$. To be precise, we can see from the analysis in section 1.2 that :

$$HH_{\mathbb{Z}/2\mathbb{Z}}^*(C_*(\Omega S^2)) \cong \frac{\mathbb{C}[u, a] \otimes \Lambda b}{(a^2, ab, au)}$$

$$HH_{\mathbb{Z}/2\mathbb{Z}}^*(C^*(S^2)) \cong \frac{\mathbb{C}[[u]][a] \otimes \Lambda b}{(a^2, ab, au)}$$

For both deformed categories mentioned above, $End(\mathbb{C}((t)), \mathbb{C}((t)))$ are the same. Geometrically, the equivariant category $MF_{\mathbb{C}^*}(\tilde{X}, v)$ is mirror to the Wrapped Fukaya category consisting of exact Maslov-index zero Lagrangians.

The mirror to the category $MF(\tilde{X}, v)$ is given by considering a larger category which includes compact *weakly unobstructed* Lagrangians, such as the Lagrangian tori used to define our Lagrangian torus fibration. This category is now only $\mathbb{Z}/2\mathbb{Z}$ -graded.

The cotangent fiber is still a generator for this larger category. Following [AbouAurKat] there are exotic non-exact Lagrangian tori, contained T^*S^2 which are non-zero in the Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and which are Floer theoretically disjoint from the zero-section. These correspond to the other objects of the deformed category $MF(\tilde{X}, v + u/w)$

We have the following calculation:

Lemma 9.12. $(u/w)t \cong ut + u^3t^2$ as Maurer-Cartan classes in $HH^*(MF(\tilde{X}, v))((t))$

In view of the previous discussion it is interesting to discuss the root stacks $\mathbb{C}P^1 \times \mathbb{C}P_{(D,d)}^1$ as well. For generic t , the LG-models $(\tilde{X}, v + t(u/w)^d)$ all have isolated critical locus. Let Z denote the scheme theoretic critical locus of this function.

Recall that the orbifold cohomology $H^*(X_{(\mathcal{D},d)} \times_{(X_{(\mathcal{D},d)} \times X_{(\mathcal{D},d)})} X_{(\mathcal{D},d)})$ is the space of states for orbifold Gromov-Witten theory.

Calculating the orbifold cohomology of $X_{(\mathcal{D},d)}$ we have the following lemma:

Lemma 9.13. $\dim H_{orb}(\mathbb{C}P^1 \times \mathbb{C}P_{(D,d)}^1) = \text{length}(Z)$

This observation should imply homological mirror symmetry for the orbifold once the appropriate definitions for the entire Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1_{(D,d)}$ are in place.

More precisely, the above lemma appears to be the shadow of a homomorphism:

$$H^*(X_{(\mathcal{D},d)} \times_{X_{(\mathcal{D},d)} \times X_{(\mathcal{D},d)}} X_{(\mathcal{D},d)}) \rightarrow \mathbf{HH}^*(Fuk(X_{(\mathcal{D},d)}))$$

for some to be defined Fukaya category of the orbifold $X_{(\mathcal{D},d)}$.

In joint work with Kevin Lin, we will extend this picture to higher dimensions. For example, we examine the case of T^*S^3 . We again view T^*S^3 as

$$Y_\epsilon = \{(x, y, z, w) \in \mathbb{C}^4 : xy - zw = \epsilon\}$$

Notice that there is an action of T^2 which survives on Y_ϵ when ϵ is nonzero. Then the moment map for this T^2 action on Y_ϵ is

$$\mu_2 : (x, y, z, w) \mapsto (|x|^2 - |y|^2, |z|^2 - |w|^2).$$

Let δ be a constant; then in Y_ϵ , we have the anticanonical divisor D_δ which is given by the locus $\{xy = \delta\}$. Note that the divisor is preserved by the natural T^2 action. Letting $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$, we put

$$L_{r,\vec{\lambda}} = \{(x, y, z, w) \in Y_\epsilon : |xy - \delta| = r, \mu_2(x, y, z, w) = \vec{\lambda}\}.$$

One can show that these $L_{r,\vec{\lambda}}$, when they are smooth, are special Lagrangian tori in Y_ϵ . The mirror is the small resolution of the singularity:

$$uv = (1 + w_1)(1 + w_2)$$

with potential $w = v$.

We consider as before the variety, Q_3 , the projective quadric in $\mathbb{C}P^4$, which is again a compactification of Y_ϵ . The compactification of the affine quadric in projective space corresponds to the deformation

$$w = v + t(u^2/w_1w_2)$$

We will also generalize the above discussion and discuss the Fukaya category of the root stacks in this case.

10. MIRROR SYMMETRY FOR THE A_n PLUMBINGS AND CURVED DEFORMATIONS OF PLUMBINGS

Previously, we have discussed the simplest sort of Stein manifold, cotangent bundles. In this section, we look at the next simplest case, which is the A_n plumbing of cotangent bundles of S^2 . Here the situation is considerably more complicated, and there is no known homotopical description

of the wrapped Fukaya category. In this section, we compute the category for plumbings of spheres and then examine its curved deformation theory.

Again we consider the A_n plumbing of spheres as a hypersurface:

$$Y_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{n+1} = \epsilon\}$$

Theorem 10.1. $WFuk(Y_\epsilon) \cong MF_{\mathbb{C}^*}(\tilde{X}, v)$

The space (\tilde{X}, v) is the mirror of Y_ϵ , whose construction follows closely the ideas of the previous section. More precisely, the map

$$z : Y_\epsilon \rightarrow \mathbb{C}$$

defines a Lefschetz fibration.

Furthermore, we have as before an action of S^1 on this manifold given by

$$\alpha * (x, y, z) = (\alpha x, \alpha^{-1}y, z)$$

One can then construct a special Lagrangian fibration on Y_ϵ using the same recipe as before. For a detailed exposition of how to construct the mirror using this special Lagrangian fibration, the reader may consult [Chan], based upon an earlier talk of Auroux [?].

We again consider the minimal resolution \tilde{X} of the A_n singularity,

$$X = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* : uv - (1 + w)^{n+1} = 0\}$$

The mirror is best described as an open part of a toric variety. Let

$$X_{tor} = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C} : uv - (1 + w)^{n+1} = 0\}$$

The fan for this toric singularity is given by Δ in $N_{\mathbb{R}}$ for $N = \mathbb{Z}^2$ consisting of the cone generated by the vectors:

$$v_{\rho_1} = (0, 1), v_{\rho_2} = (n, 1),$$

The toric resolution is given by the fan Δ in $N_{\mathbb{R}}$ for $N = \mathbb{Z}^2$ consisting of all cones generated by no more than two of the vectors:

$$v_{\rho_1} = (0, 1), v_{\rho_2} = (1, 1), v_{\rho_3} = (2, 1), \dots, v_{\rho_{n+1}} = (n, 1)$$

The mirror is equipped with a natural \mathbb{C}^* action:

$$\alpha * (u, v, w) = (\alpha^2 u, \alpha^{-2} v, w)$$

It is with respect to this action that we consider the category of graded D-Branes. The zero fiber of v is an A_n configuration of \mathbb{CP}^1 in the fiber of the resolution over $(0, 0, -1)$ in $\tilde{X} \rightarrow X$, the components of which we denote by Z_i , $i = 1, \dots, n$ glued to an \mathbb{A}^1 , which we denote by A at a single point along Z_1 . The reduced scheme structure on the singular locus of v coincides with $Z_i \cup A$ for $i = 1, \dots, n-1$. Under the mirror map, each \mathbb{CP}^1 in the zero fiber corresponds to a matching sphere Lagrangian and the \mathbb{A}^1 corresponds to a Lefschetz thimble at the beginning of the chain.

We will need to study the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$. We recall its definition here. Objects consist of (twisted complexes of) exact Lagrangians in Y_ϵ and an ordered collection of Lefschetz thimbles $\Delta_1, \Delta_2, \dots, \Delta_{n+1}$. Let V_1, V_2, \dots, V_{n+1} denote the corresponding vanishing cycles. In his book [Sei2], Seidel defines the Floer cohomology for the thimbles as:

$$Hom(\Delta_i, \Delta_j) = \begin{cases} \mathbb{C}, & i = j \\ HF^*(V_i, V_j), & i < j \\ 0, & i > j \end{cases}$$

We apply the following theorem of Abouzaid and Seidel, which enables us to compute the wrapped Fukaya category of the Lefschetz fibration in terms of the inclusion of a smooth fibre and the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$ [Abou2, Sei4] :

Theorem 10.2. *There is a natural transformation from Serre: $[Fuk(Y_\epsilon, z)] \rightarrow [Fuk(Y_\epsilon, z)]$ to the $id : [Fuk(Y_\epsilon, z)] \rightarrow [Fuk(Y_\epsilon, z)]$ such that $WFuk(Y_\epsilon)$ is isomorphic to the localization of $Fuk(Y_\epsilon, z)$ with respect to this natural transformation.*

We describe how this works. We denote by B the exceptional algebra associated to the thimbles in $Fuk(Y_\epsilon, z)$. Let E be the algebra associated to the vanishing cycles in the fiber $Fuk(Y_z)$. We have an inclusion $B \rightarrow E$ of A_∞ algebras. We have an exact sequence of $B - B$ bimodules

$$B \rightarrow E \rightarrow E/B$$

taking the boundary morphism $E/B \rightarrow B$ gives rise to the above natural transformation.

The strategy we will follow is to first produce a mirror to $[Fuk(Y_\epsilon, z)]$ and then use the above theorem of Seidel and Abouzaid to deduce the result. We prove the result by partially compactifying \tilde{X} by adding an extra divisor D , which corresponds to adding the vector $(-1, 0)$ to the above toric fan.

This is mirror to the Lefschetz fibration (Y_ϵ, z) . Notice that the divisor D is isomorphic to \mathbb{C}^* which is mirror to the fibre $Y_z \rightarrow Y_\epsilon$ (also a \mathbb{C}^*).

We denote this compactified space by \overline{X} . Notice that the divisor D is an anti-canonical divisor for \overline{X} . The singular locus of the potential v when extended to the space \overline{X} is a chain of \mathbb{CP}^1 , whose irreducible components consist of the compactification of A denoted by \bar{A} and Z_i .

Let L denote $\mathcal{O}_{\bar{A}} \oplus_i \mathcal{O}_{Z_i}$, $i = 1, \dots, n$ as objects in $MF(\overline{X}, v)$. The object L generates the category because it is proven in [LinPom] that a generator of the category of coherent sheaves on the singular locus generates the category $MF(\overline{X}, v)$. Picking a point p away from the singular locus on Z_n , we have that the collection $p \oplus \mathcal{O}_{Z_n}$ generates $Coh(Z_n)$. In particular, it generates the skyscraper sheaf at the point $q = Z_n \cap Z_{n-1}$. Inductively, we can now generate the category $Coh(\cup Z_i)$, where $i = 1, \dots, n-1$. The object $[p]$ is zero in $MF(\overline{X}, v)$, since it avoids the critical locus.

It is interesting to notice that in the case of $MF(\tilde{X}, v)$, the category of matrix factorizations is actually generated by $\mathcal{O}_A \oplus_i \mathcal{O}_{Z_i}$ $i = 1, \dots, n-1$. The reason is that A is affine so we can generate the skyscraper sheaf at $q' = A \cap Z_1$. Again using an inductive argument, we can generate the category of coherent sheaves on the entire critical locus.

Similarly, let \mathcal{L} denote the sum of Δ_1 and the matching spheres \mathcal{L}_j . Seidel proves that the Lefschetz thimbles split generate the Fukaya-Seidel category [Sei2]. As noted above, in our setting it is more appropriate to consider a different generating set for the Fukaya category. The thimble Δ_1 and the matching spheres \mathcal{L}_j also generate the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$. This can be seen because given a thimble at the beginning of the chain and all of the matching spheres, the other thimbles arise as a Dehn twist [Sei2] of the preceeding thimble and the matching sphere.

It is easy to verify on the homology level that:

$$Hom_{MF(\overline{X}, v)}(L, L) \cong Hom_{Fuk(Y_\epsilon, z)}(\mathcal{L}, \mathcal{L})$$

The homology $Hom_{MF(\overline{X}, v)}(L, L)$ has the following quiver presentation.

$$Q = (\alpha_0 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} \alpha_1 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} \alpha_2 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} \cdots \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} \alpha_{n-1} \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} \alpha_n)$$

The quiver is given by taking the path ring of the above graph modulo the following relations

$$(\alpha_0 | \alpha_1 | \alpha_0) = 0, (\alpha_i | \alpha_{i+1} | \alpha_i) = (\alpha_i | \alpha_{i-1} | \alpha_i)$$

and

$$(\alpha_{i-1} | \alpha_i | \alpha_{i+1}) = (\alpha_{i+1} | \alpha_i | \alpha_{i-1}) = 0$$

Our grading conventions follow those of the foundational [SeiTho]. We next explain how to verify this. The calculation of $Hom_{MF(\overline{X}, v)}(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})$ proceeds by analogy with the corresponding calculation in the ordinary derived category. As was proven in [LinPom], we can compute $Hom(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i})$

by $R\Gamma(R\mathcal{H}om(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}))$. The local calculation in the previous section demonstrates that

$$R\mathcal{H}om(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) \cong \mathcal{O}_{Z_i} \oplus \mathcal{N}_{Z_i}$$

. Thus $Hom^0(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) \cong \mathbb{C}$.

The fact that the branes \mathcal{O}_{Z_i} do not intersect the compactifying divisor implies that:

$$Hom^2(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_i}) \cong \mathbb{C}$$

For \mathcal{O}_L , $\mathcal{N}_L \cong \mathcal{O}(-1)$ and $Hom^2(\mathcal{O}_L, \mathcal{O}_L)$ vanishes. The calculations that

$$Hom_{MF(\overline{X}, v)}(\mathcal{O}_{Z_i}, \mathcal{O}_{Z_{i+1}}) \cong \mathbb{C}$$

and

$$Hom_{MF(\overline{X}, v)}(\mathcal{O}_{Z_{i+1}}, \mathcal{O}_{Z_i}) \cong \mathbb{C}$$

are also purely local. If the natural equivariant structure on each of these objects is taken into account, the gradings also agree with those listed above.

We also have the following formality lemma for the $Hom_{MF(\overline{X}, v)}(L, L)$ in the matrix factorization category, whose proof is an adaptation of an argument of Seidel and Thomas [SeiTho]:

Lemma 10.3. *The quiver algebra $Hom_{MF(\overline{X}, v)}(L, L)$ is intrinsically formal.*

We summarize the main point. Lemma 4.21 of the paper by Seidel and Thomas prove that the A_{n+1} quiver algebra, which we denote by A is formal. We denote the quiver that we care about by \tilde{A} . For degree reasons we have the exact sequence:

$$\mathbf{HH}^{q-1}(A, A[2-q]) \rightarrow \mathbf{HH}^q(A, A[2-q]) \rightarrow 0$$

Using their notation, we define $\phi_{i_0, i_1, \dots, i_0} \in \mathbf{HH}^q(A, A[2-q])$ to be

$$\phi_{i_0, i_1, \dots, i_0}(c) = \begin{cases} (i_0|i_0 + 1|i_0), & c = (i_0|i_1) \otimes \dots \otimes (i_{q-1}|i_0) \\ 0, & \text{all other basis elements } c \end{cases}$$

These form a basis for $\mathbf{HH}^q(A, A[2-q])$. Seidel and Thomas use two sets of classes of Hochschild cochains in $\mathbf{HH}^{q-1}(A, A[2-q])$ to generate enough relations in $HH^q(A, A[2-q])$ to prove that this group is zero. These classes are:

$$\begin{aligned} \phi'(c) &= \begin{cases} (i_0|i_{q-2}), & c = (i_0|i_1) \otimes (i_2|i_3) \dots \otimes (i_{q-3}|i_{q-2}) \\ 0, & \text{all other basis elements } c \end{cases} \\ \phi''(c) &= \begin{cases} (i_0|i_1|i_0), & c = (i_0|i_1|i_0) \otimes (i_3|i_4) \dots \otimes (i_{q-2}|i_0) \\ 0, & \text{all other basis elements } c \end{cases} \end{aligned}$$

The first set of classes are unaffected by annihilating $(\alpha_0|\alpha_1|\alpha_0)$. The second set of classes are precisely those classes which annihilate $\phi_{\alpha_0, \alpha_1, \dots, \alpha_0}$, which also vanish when we annihilate $(\alpha_0|\alpha_1|\alpha_0)$. Thus, the same argument implies that $HH^2(\tilde{A}, \tilde{A}) = 0$.

It follows that after idempotent completion, $Fuk(Y_\epsilon, z) \cong MF(\overline{X}, v)$. From here, we note that the divisor D defines a section of the anti-canonical line bundle \mathcal{K}^{-1} on \overline{X} and hence a natural transformation. This natural transformation is given by taking a matrix factorization P , tensoring it with \mathcal{K} , and using the section defining our divisor \mathcal{D} to define a morphism:

$$P \otimes \mathcal{K} \rightarrow P$$

Lemma 10.4. *$MF(\tilde{X}, v)$ is the localization of $MF(\overline{X}, v)$ with respect to this natural transformation. This natural transformation is mirror to that induced by the inclusion of the fibre $Y_z \subset Y_\epsilon$.*

To begin, we observe that the functor

$$\pi : MF(\overline{X}, v) \rightarrow MF(\tilde{X}, v)$$

is essentially surjective because coherent sheaves extend from open subsets. Next, let \mathcal{C} be a category, F a functor and $N : F \rightarrow id$ a natural transformation. Let π denote the functor

$$\mathcal{C} \rightarrow \mathcal{C}_{loc}$$

where \mathcal{C}_{loc} denotes the localization with respect to N . Then we have that:

$$\varinjlim Hom_{\mathcal{C}}(F^p(X), Y) \cong Hom_{\mathcal{C}_{loc}}(\pi(X), \pi(Y))$$

Recall that given two matrix factorizations, we can compute $Hom(E, F)$ by $R\Gamma(Hom(E, F))$. The result now follows since:

$$R\Gamma(Hom(E, F)|_{\tilde{X}}) \cong \varinjlim R\Gamma(Hom(E, F \otimes K^{-p}))$$

To show that the natural transformations are equivalent, note that the natural transformations are trivial restricted to the \mathcal{O}_{Z_i} . Thus, we only have to consider the object $\mathcal{O}_{\tilde{A}}$. By the matrix factorization calculations mentioned above, we have $Hom^0(\mathcal{O}_{\tilde{A}}, \mathcal{O}_{\tilde{A}}(1)) \cong \Gamma(\mathcal{O}_{\tilde{A}}(1))_0 \cong \mathbb{C}$. Here $\Gamma(\mathcal{O}_{\tilde{A}}(1))_0$ denotes the invariant part of $\mathcal{O}_{\tilde{A}}(1)$ with respect to the natural induced equivariant structure on $\mathcal{O}_{\tilde{A}}(1)$. From this information it is easy to conclude that the two natural transformations give rise to isomorphic categories after localization.

To demonstrate the non-trivial nature of the wrapped Fukaya category of the plumbing, one can compute directly the endomorphisms of the Lefschetz thimble in the wrapped category. For example, the thimble at the beginning of the chain discussed above has endomorphism algebra:

$$\mathbb{C}[u] \otimes \Lambda(e) \quad m_{n+1}(e, e, e, \dots, e) = u^n$$

We conclude our thesis by stating the following proposition, which the reader can check by direct computation.

Lemma 10.5. *The curved deformations corresponding to u^j compactify the category.*

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